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SPHEROIDAL DECOMPOSITIONS OF E^4(1)

BY
J. P. NEUZIL

Abstract. This paper investigates a generalization to E^4 of the notion of toroidal decomposition of E^3. A certain type of this kind of upper semicontinuous decomposition is shown to be shrinkable and hence yield E^4 as its decomposition space.

1. **Introduction.** In this paper we investigate a generalization to E^4 of the notion of toroidal decomposition of E^3. The following is a common method of constructing an upper semicontinuous decomposition of E^n: Let $M_0, M_1, M_2, \dots$ be compact n-manifolds-with-boundary in E^n such that for each i, $M_{i+1} \subset \text{Int } M_i$. Then let the nondegenerate elements of G be the nondegenerate components of $\bigcap \{M_i : i \geqq 0\}$. If $n=3$ and each component of each M_i is a solid torus then G is a toroidal decomposition. Examples of point-like toroidal decompositions G of E^3 such that E^3/G is not homeomorphic to E^3 have been given by Bing [3], Sher [6], and Bing and Armentrout [1].

We shall say a decomposition G of E^4 is a spheroidal decomposition if it is constructed in the manner described above and each component of each M_i is homeomorphic to $S^2 \times D^2$. Our result is that if G is a point-like spheroidal decomposition of E^4 such that the components of the manifolds used in the construction have some simple unknotting properties then E^4/G is homeomorphic to E^4. One corollary is the following. Suppose G is a point-like spheroidal decomposition of E^4 such that if X is a component of M_i then $M_{i+1} \cap X$ has exactly two components, say $S_1 \times D_1$ and $S_2 \times D_2$, and each $S_j \times \{0\}$ is unknotted in X. Then E^4/G is homeomorphic to E^4. Sher has shown that the corresponding statement for toroidal decompositions of E^3 is not true [6]. A special case of the result of this paper has been done by Lininger [5].

2. **Notation and terminology.** If A is a subset of a metric space, $N_\varepsilon(A)$ will denote the open ε-neighborhood of A. S^n will denote the unit sphere in E^{n+1}, D^n the closed unit disk in E^n, and I^k the product of $[-1, 1]$ with itself k times. We shall use the terminology of piecewise linear topology as found in [4]. All manifolds

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(1) This paper is essentially the author's doctoral dissertation which was written under the supervision of Professor Steve Armentrout at the University of Iowa, Iowa City.

embedded in E^n will be assumed to be polyhedral. If X is a polyhedral k-sphere (or k-disk) in E^n then X is *unknotted* if there exists a piecewise linear homeomorphism of E^n onto itself which takes X onto Bd I^{k+1} (or I^k). If B^{n-1} is an $(n-1)$-cell in an n-cell B^n then (B^n, B^{n-1}) is a *standard cell pair* if it is piecewise linearly homeomorphic to the pair

$$([-1, 1]^n, [-1, 1]^{n-1} \times \{0\}).$$

If G is an upper semicontinuous decomposition then H_G will denote the union of the nondegenerate elements of G.

If X and Y are topological spaces then a function h from $X \times [0, 1]$ to Y is a *homotopy*. h_t denotes the function from X to Y defined by $h_t(x) = h(x, t)$. If each h_t is a homeomorphism then the homotopy is called an *isotopy*. If h_t is a homeomorphism for $t < 1$ then the homotopy is called a *pseudo-isotopy*. In the remainder of the paper a homotopy will be denoted by h_t. If h_t and g_t are isotopies from X into itself and g_0 is the identity map on X we define $g_t * h_t$ to be the isotopy defined by

$$g_t * h_t(x) = h_{2t}(x) \qquad \text{if } 0 \leqq t \leqq \tfrac{1}{2},$$

and

$$g_t * h_t(x) = g_{2t-1} \circ h_1(x) \quad \text{if } \tfrac{1}{2} \leqq t \leqq 1.$$

For any space id will denote the identity map on that space. If M is a triangulated manifold-with-boundary and h_t is an isotopy on M (that is from M onto itself) then h_t is a *push* on M if $h_0 = \text{id}$, $h_t | \text{Bd } M = \text{id}$ for all t, and each h_t is piecewise linear.

A cellular decomposition G of E^4 is *spheroidal* if there exists a sequence $M_0, M_1, \ldots$ of compact polyhedral 4-manifolds-with-boundary in E^4 such that (1) for each i, $M_{i+1} \subset \text{Int } M_i$ and each component of M_i is piecewise linearly homeomorphic to $S^2 \times D^2$ and (2) the nondegenerate elements of G are the nondegenerate components of $\bigcap \{M_i : i \geqq 0\}$. A spheroidal decomposition G of E^4 is *simple* if whenever X is a component of M_i and $X_1, \ldots, X_n$ are the components of M_{i+1} in X then (1) there is an integer k and a piecewise linear homeomorphism f of $S^2 \times D^2$ onto X_k such that $f(S^2 \times \{0\})$ bounds an unknotted polyhedral 3-cell in Int X and (2) there is a polyhedral 4-cell in Int X containing

$$\bigcup \{X_i : 1 \leqq i \leqq n \text{ and } i \neq k\}.$$

Let G be a spheroidal decomposition of E^4. We shall adopt the following notation to describe the 4-manifolds-with-boundary $M_0, M_1, \ldots$ which define the elements of G. We shall assume $M_0 = X_0 = S_0 \times D_0$ where S_0 is a 2-sphere and D_0 is a 2-cell. The components of M_1 will be $X_1, \ldots, X_{m_0}$ where $X_i = S_i \times D_i$. If j is a positive integer, the components of M_j will be $X_{i_1 i_2 \ldots i_j} = S_{i_1 i_2 \ldots i_j} \times D_{i_1 i_2 \ldots i_j}$ where for certain positive integers $m(0), m(i_1), m(i_1 i_2), \ldots,$ and $m(i_1 i_2 \ldots i_{j-1})$, we have $1 \leqq i_1 \leqq m(0)$, $1 \leqq i_2 \leqq m(i_1)$, $1 \leqq i_3 \leqq m(i_1 i_2), \ldots,$ and $1 \leqq i_j \leqq m(i_1 i_2 \ldots i_{j-1})$. Then the components of M_{j+1} in $X_{i_1 i_2 \ldots i_j}$ will be denoted by $X_{i_1 i_2 \ldots i_j 1}, X_{i_1 i_2 \ldots i_j 2}, \ldots,$ and

$X_{i_1 i_2 \ldots i_j m(i_1 i_2 \ldots i_j)}$ where $X_{i_1 i_2 \ldots i_j k} = S_{i_1 i_2 \ldots i_j k} \times D_{i_1 i_2 \ldots i_j k}$ where, for each k, $S_{i_1 i_2 \ldots i_j k}$ is a 2-sphere and $D_{i_1 i_2 \ldots i_j k}$ is a 2-cell.

The statement α is an index means either $\alpha = 0$ or for some positive integer n, $\alpha = i_1 i_2 \ldots i_n$ where $1 \leqq i_1 \leqq m(0)$, and for $k = 2, 3, \ldots$, or n, $1 \leqq i_k \leqq m(i_1 i_2 \ldots i_{k-1})$. If α is the index $i_1 i_2 \ldots i_n$ and $1 \leqq i \leqq m(i_1 i_2 \ldots i_n)$ then αi denotes the index $i_1 i_2 \ldots i_n i$. An index $\alpha = i_1 i_2 \ldots i_n$ will be called a stage n index. Hence, if α is a stage n index, X_α is a component of M_n and $M_{n+1} \cap X_\alpha = \bigcup \{X_{\alpha i} : 1 \leqq i \leqq m(\alpha)\}$. We shall let 0 denote the center point of each D_α. Thus $S_\alpha \times \{0\}$ is a spine of X_α.

Next we describe a coordinatization of S^2 and name some subsets of S^2. We shall consider E^2 to have polar coordinates. Let $\tilde{D}^2 = \{(\rho, \theta) \in E^2 : 0 \leqq \rho \leqq 2\}$. There is a map Φ from $\tilde{D}^2$ onto S^2 such that Φ is a homeomorphism on Int $\tilde{D}^2$ and $\Phi(\text{Bd } \tilde{D}^2)$ is a single point. Then Φ gives a (polar) coordinatization of S^2 in terms of the polar coordinates of $\tilde{D}^2$. Henceforth we shall use (ρ, θ) to denote points of S^2 in this coordinatization. Therefore if (ρ, θ) and (ρ', θ') are points of S^2 then $(\rho, \theta) = (\rho', \theta')$ if and only if $\rho = \rho'$ and $\theta = \theta'$ (mod 2π), or $\rho = \rho' = 2$.

Now if $0 \leqq t_1 < t_2 \leqq 2$ we let

$$A(t_1, t_2) = \{(\rho, \theta) \in S^2 : t_1 \leqq \rho \leqq t_2\}.$$

Then $A(t_1, t_2)$ is an annulus if $0 < t_1 < t_2 < 2$ and $A(t_1, t_2)$ is a disk if either $0 < t_1 < t_2 = 2$ or $0 = t_1 < t_2 < 2$. $A(0, 2) = S^2$. See Figure 1a. If $0 \leqq t \leqq 2$, we let

$$C(t) = \{(\rho, \theta) \in S^2 : \rho = t\}.$$

$C(t)$ is a circle if $0 < t < 2$ but degenerates to a point if $t = 0$ or $t = 2$. If $a_1 < a_2$ we let

$$Z(a_1, a_2) = \{(\rho, \theta) \in S^2 : a_1 \leqq \theta \leqq a_2\}.$$

If a is any number we let

$$M(a) = \{(\rho, \theta) \in S^2 : \theta = a\}.$$

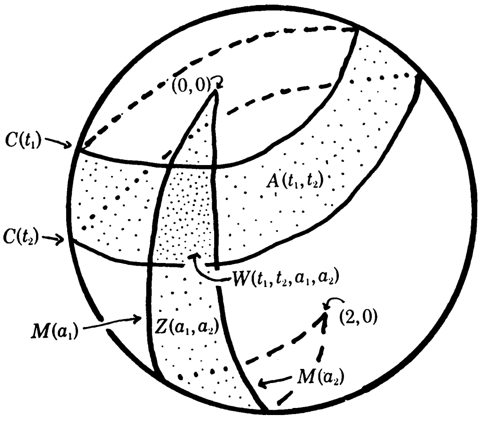

FIGURE 1a

If we think of (0, 0) and (2, 0) as being the poles of S^2 then each $C(t)$ is a parallel of latitude on S^2 and each $M(a)$ is a meridian of longitude. If $0 \leqq t_1 < t_2 \leqq 2$ and $a_1 < a_2$ then $W(t_1, t_2, a_1, a_2)$ will denote $A(t_1, t_2) \cap Z(a_1, a_2)$. If $X(\cdot)$ is any of the sets defined above then we let $X^*(\cdot) = X(\cdot) \times D^2$. For instance,

$$A^*(t_1, t_2) = A(t_1, t_2) \times D^2.$$

Next we describe some regular neighborhoods of the sets just defined.

If $0 < t_1 - e < t_1 < t_2 < t_2 + e < 2$ then $R_e(A(t_1, t_2)) = A(t_1 - e, t_2 + e)$.

If $0 < t_2 < t_2 + e < 2$ then $R_e(A(0, t_2)) = A(0, t_2 + e)$.

If $0 < t_1 - e < t_1 < 2$ then $R_e(A(t_1, 2)) = A(t_1 - e, 2)$.

If $a_1 - e < a_1 < a_2 < a_2 + e$ and $|(a_2 + e) - (a_1 - e)| < 2\pi$, then

$$R_e(Z(a_1, a_2)) = Z(a_1 - e, a_2 + e) \cup A(0, e) \cup A(2 - e, 2).$$

See Figure 1b.

If $R_e(A(t_1, t_2))$ and $R_e(Z(a_1, a_2))$ are both defined then

$$R_e(W(t_1, t_2, a_1, a_2)) = R_e(A(t_1, t_2)) \cap R_e(Z(a_1, a_2)).$$

Also we let $R_e(A^*(t_1, t_2)) = R_e(A(t_1, t_2)) \times D^2$, $R_e(Z^*(a_1, a_2)) = R_e(Z(a_1, a_2)) \times D^2$, and $R_e(W^*(t_1, t_2, a_1, a_2)) = R_e(W(t_1, t_2, a_1, a_2)) \times D^2$.

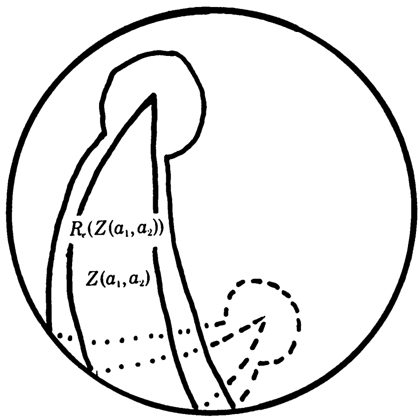

FIGURE 1b

3. The main result.

THEOREM. *If G is a simple spheroidal decomposition of E^4, U is an open set containing H_G and $\varepsilon > 0$ then there exists a push h_t on E^4 such that* (1) $h_t | E^4 - U = \text{id}$ *for all t and* (2) *if $g \in G$ then* $\text{diam}\, h_1(g) < \varepsilon$.

By the proof of Theorem 1 of [2], the following is an immediate consequence.

COROLLARY. *If G is a simple spheroidal decomposition of E^4 then E^4/G is homeomorphic to E^4.*

The main tool in the proof of the theorem is Proposition (r, s) which will be proved in the following sections. We first state Proposition (r, s) and give a proof of the theorem. Throughout the rest of the paper we assume G is a fixed simple spheroidal decomposition of E^4 and we shall use the notation of §2 to describe the manifolds-with-boundary which define G.

PROPOSITION (r, s). *Let F be a piecewise linear homeomorphism of $S_0 \times D_0 = X_0$ onto $S^2 \times D^2$. Suppose $0 = t_0 < \cdots < t_r < t_{r+1} = 2$, $0 = a_0 < \cdots < a_s < a_{s+1} = 2\pi$ and $e > 0$. Then there exists a push h_t on $S^2 \times D^2$ and an integer n such that, for each stage n index α,*

$$h_1(F(X_\alpha)) \subset \operatorname{Int} R_e(W^*(t_{i-1}, t_i, a_{j-1}, a_j))$$

for some $i = 1, \ldots,$ or $r+1$ and some $j = 1, \ldots,$ or $s+1$.

Proof of the theorem. Let $H_G(\varepsilon) = \bigcup \{g \in G : \operatorname{diam} g \geqq \varepsilon\}$. Since $H_G(\varepsilon)$ and $E^n - U$ are disjoint closed sets, there is an integer n' such that if X_α is a component of $M_{n'}$ intersecting $H_G(\varepsilon)$ then $X_\alpha \subset U$. (If not then some element of G would intersect both $E^n - U$ and $H_G(\varepsilon)$.) Let

$$\mathfrak{A} = \{\alpha : \alpha \text{ is a stage } n' \text{ index and } X_\alpha \cap H_G(\varepsilon) \neq \varnothing\}.$$

For each α in $\mathfrak{A}$ let F_α be a piecewise linear homeomorphism of X_α onto $S^2 \times D^2$. We may choose numbers $t_0, \ldots, t_{r+1}, a_0, \ldots, a_{s+1}$ and e which satisfy the hypothesis of Proposition (r, s) and a disk $\tilde{D}^2 \subset D^2$ such that, for each $\alpha \in \mathfrak{A}$ and each i and j,

$$\operatorname{diam} F_\alpha^{-1}(R_e(W(t_{i-1}, t_i, a_{j-1}, a_j) \times \tilde{D}^2)) < \varepsilon.$$

By Proposition (r, s), for each $\alpha \in \mathfrak{A}$ there is an integer n_α and a push h_t^α on $S^2 \times D^2$ such that if β is a stage $(n' + n_\alpha)$ index such that $X_\beta \subset X_\alpha$ then

$$h_1^\alpha(F_\alpha(X_\beta)) \subset \operatorname{Int} R_e(W^*(t_{i-1}, t_i, a_{j-1}, a_j))$$

for some $i = 1, \ldots,$ or $r+1$ and some $j = 1, \ldots,$ or $s+1$. Let g_t be a push on $S^2 \times D^2$ such that

(1) for each $x \in S^2$, $g_t(\{x\} \times D^2) = \{x\} \times D^2$ for all t, and

(2) for each $\alpha \in \mathfrak{A}$, $g_1 \circ h_1^\alpha(F_\alpha(X_\alpha \cap M_{n'+n_\alpha})) \subset \operatorname{Int}(S^2 \times \tilde{D}^2)$.

Then for each $\alpha \in \mathfrak{A}$, if β is a stage $(n' + n_\alpha)$ index and $X_\beta \subset X_\alpha$, then $g_1 \circ h_1^\alpha(F_\alpha(X_\beta))$ is contained in $\operatorname{Int}[R_e(W(t_{i-1}, t_i, a_{j-1}, a_j)) \times \tilde{D}^2]$ for some $i = 1, \ldots,$ or $r+1$ and some $j = 1, \ldots,$ or $s+1$.

Let $n = \max\{n' + n_\alpha : \alpha \in \mathfrak{A}\}$. Then h_t is defined by $h_t | X_\alpha = F_\alpha^{-1} \circ (g_t * h_t^\alpha) \circ F_\alpha$ if $\alpha \in \mathfrak{A}$ and $h_t = \mathrm{id}$ outside $\bigcup \{X_\alpha : \alpha \in \mathfrak{A}\}$.

4. **Proposition** (1, 1).

LEMMA 1. *Suppose x and y are distinct points of S^2 and $S_1, \ldots,$ and S_m are mutually disjoint polyhedral 2-spheres in* Int $(S^2 \times D^2)$ *such that* (1) *S_1 bounds an unknotted polyhedral* 3-*cell B^3 in* Int $(S^2 \times D^2)$ *and* (2) *there exists a polyhedral* 4-*cell B^4 in* Int $(S^2 \times D^2)$ *such that $S_2 \cup \cdots \cup S_m \subset \text{Int } B^4$. Then there exists a push h_t on $S^2 \times D^2$ such that $h_1(S_1) \cap \{x\} \times D^2 = \varnothing$ and $h_1(S_2 \cup \cdots \cup S_m) \cap \{y\} \times D^2 = \varnothing$.*

Proof. Let $X = \{x\} \times D^2$, $Y = \{y\} \times D^2$ and $M = S_2 \cup \cdots \cup S_m$. We may assume $M \cap (X \cup Y) = \varnothing$. This may be accomplished as follows: Select a point p in Int $(B^4) - (X \cup Y)$. Then there exists a push on $S^2 \times D^2$ which is the identity outside B^4 and which shrinks M into a neighborhood of p which is disjoint from $X \cup Y$.

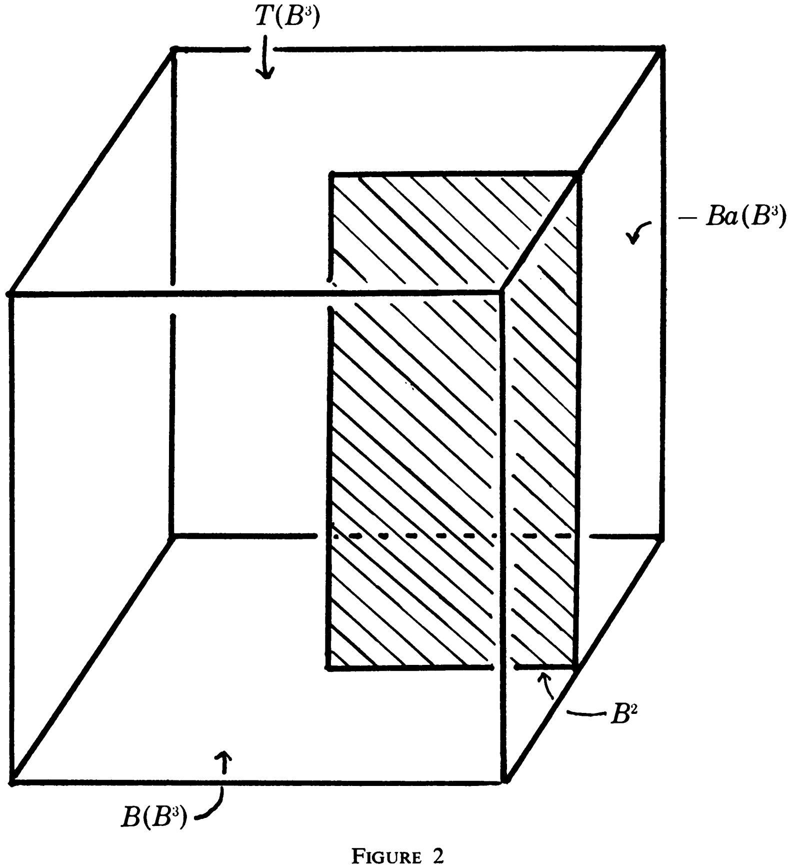

FIGURE 2

Next we construct several pushes which improve the intersection of S_1 and X and which do not push M onto Y. Initially the pushes will push X and Y instead of $S_1, S_2, \ldots, S_m$. Then the push we seek will be constructed from the inverses of these. We shall construct pushes called h_t^i for $i = 1, 2, 3,$ and 4 and g_t^i for $i = 1, 2,$ and 3. For convenience we shall write H^i for h_1^i and G^i for g_1^i.

Now, since B^3 is unknotted, there exists a piecewise linear embedding K of $[-2, 2]^4$ into Int $(S^2 \times D^2)$ such that $K([-1, 1]^3 \times \{0\}) = B^3$. Let $K([-2, 2]^3 \times \{0\}) = \tilde{B}^3$ and $K([-2, 2]^4) = \tilde{B}^4$. Let

$$T(B^3) = K([-1, 1]^2 \times \{1\} \times \{0\}),$$

$$B(B^3) = K([-1, 1]^2 \times \{-1\} \times \{0\}),$$

and

$$Ba(B^3) = K(\{-1\} \times [-1, 1]^2 \times \{0\}).$$

$T(B^3)$, $B(B^3)$ and $Ba(B^3)$ may be thought of as the top, bottom and back of B^3. See Figure 2. After a general position adjustment, each component of $(X \cup Y) \cap B^3$ is

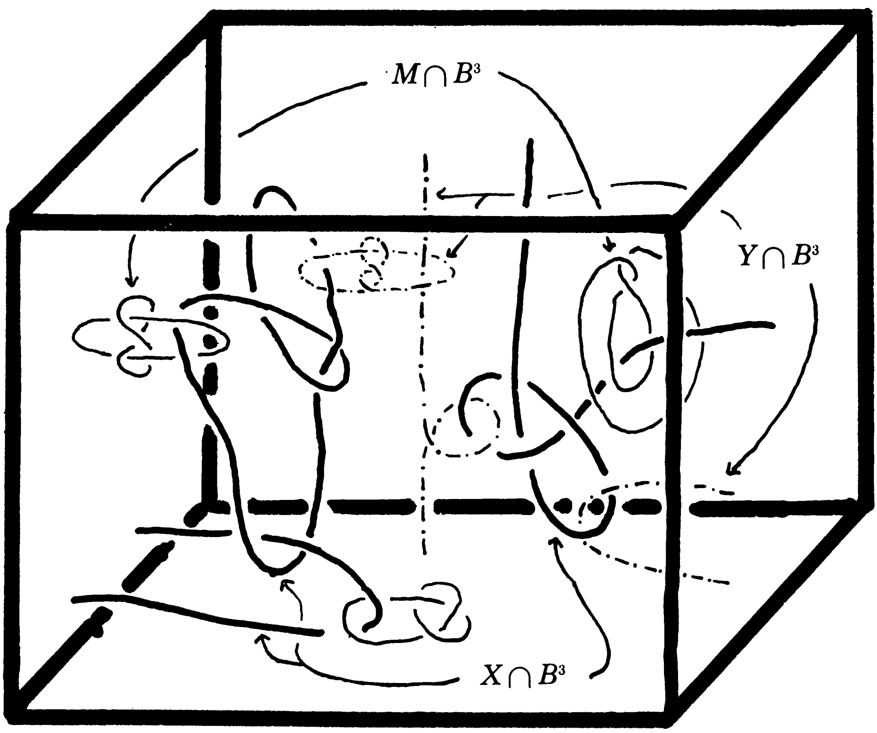

FIGURE 3

a spanning arc of B^3 or a simple closed curve in Int B^3 and each component of $M \cap B^3$ is a simple closed curve in Int B^3. See Figure 3. Now we begin to improve $X \cap B^3$.

Step 1. In this step we construct a push h_t^1 on $\tilde{B}^4$ such that

(1) $h_t^1 | (Y \cup M) = \text{id}$ and

(2) each component of $H^1(X) \cap B^3$ $(=h_1^1(X) \cap B^3)$ is a simple closed curve in Int B^3 or a spanning arc with endpoints in $T(B^3) \cup B(B^3)$.

We shall describe the construction of h_t^1 in detail. $X \cap [\text{Bd } B^3 - [T(B^3) \cup B(B^3)]]$ is a finite set $\{x_1, \ldots, x_n\}$. There exist mutually disjoint polyhedral 3-cells $B^3(1), \ldots,$ and $B^3(n)$ in Int $\tilde{B}^3$ such that for each $i = 1, \ldots,$ or n,

(3) $B^3(i) \cap B^3$ is a 3-cell,

(4) $B^3(i) \cap \text{Bd } B^3$ is a 2-cell $B^2(i)$,

(5) $B^2(i) \cap X = \{x_i\}$,

(6) $(B^3(i), B^2(i))$ is a standard cell pair,

(7) $B^3(i) \cap (M \cup Y) = \varnothing$, and

(8) $B^2(i)$ intersects Int $[T(B^3) \cup B(B^3)]$.

See Figure 4. Let $B^4(1), \ldots,$ and $B^4(n)$ be mutually disjoint polyhedral 4-cells in

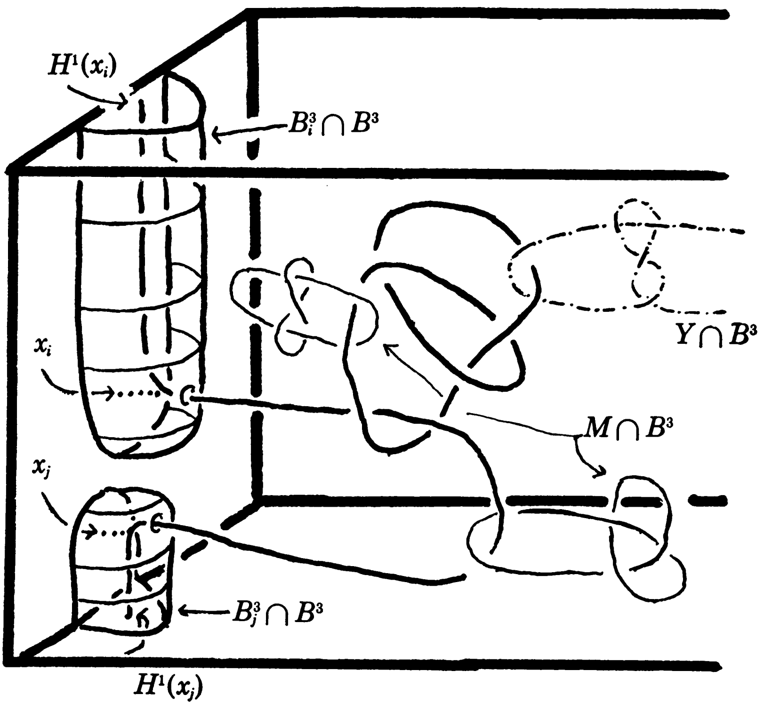

FIGURE 4

$\tilde{B}^4$ such that for each $i = 1, \ldots,$ or n,

(9) $B^4(i) \cap \tilde{B}^3 = B^3(i)$,

(10) $B^4(i) \cap (M \cup Y) = \varnothing$ and

(11) $(B^4(i), B^3(i))$ is a standard cell pair.

Then, for each $i = 1, \ldots,$ or n, there exists a push f_t^i on $B^2(i)$ such that $f_1^i(x_i) \in T(B^3) \cup B(B^3)$. Since $(B^3(i), B^2(i))$ is a standard cell pair, f_t^i may be extended to a push, still called f_t^i, on $B^3(i)$ such that $f_t^i(B^3(i) \cap B^3) = B^3(i) \cap B^3$ for all i. Then f_t^i may be extended to a push, again called f_t^i, on $B^4(i)$. Then the push desired in Step 1 is h_t^1 defined by $h_t^1 | B^4(i) = f_t^i$ and $h_t^1 = \text{id}$ elsewhere.

Step 2. Using methods similar to those of Step 1, we may construct a push h_t^2 on $\tilde{B}^4$ such that

(12) $h_t^2 | (Y \cup M) = \text{id}$ and

(13) each component of $H^2 \circ H^1(X) \cap B^3$ $(= h_1^2 \circ h_1^1(X) \cap B^3)$ is a spanning arc with endpoints in $T(B^3) \cup B(B^3)$.

Figure 5 illustrates the action of H^2 on one component of $H^1(X) \cap B^3$ which is a simple closed curve. Figure 6 illustrates the result of $H^2 \circ H^1$. (Compare with Figure 3.)

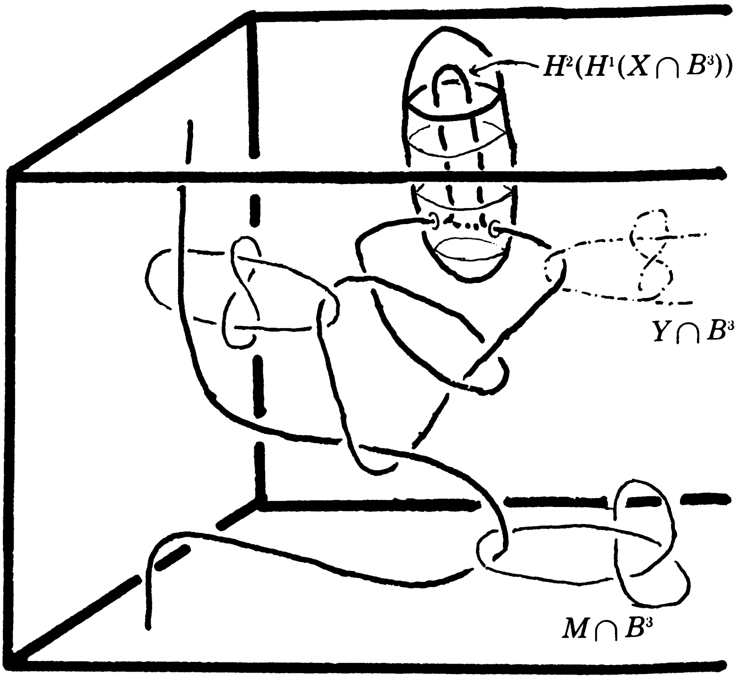

FIGURE 5

Let pr denote the projection of B^3 onto $Ba(B^3)$.

Step 3. There is a push h_t^3 on $\tilde{B}^4$ such that

(14) $h_t^3 | Y \cup M = \text{id}$,

(15) each component of $H^3 \circ H^2 \circ H^1(X) \cap B^3$ is a spanning arc with endpoints in $T(B^3) \cup B(B^3)$, and

(16) $\text{pr} | H^3 \circ H^2 \circ H^1(X) \cap B^3$ is a homeomorphism.

Figures 7a, 7b, and 7c illustrate the action of h_t^3 in successive stages.

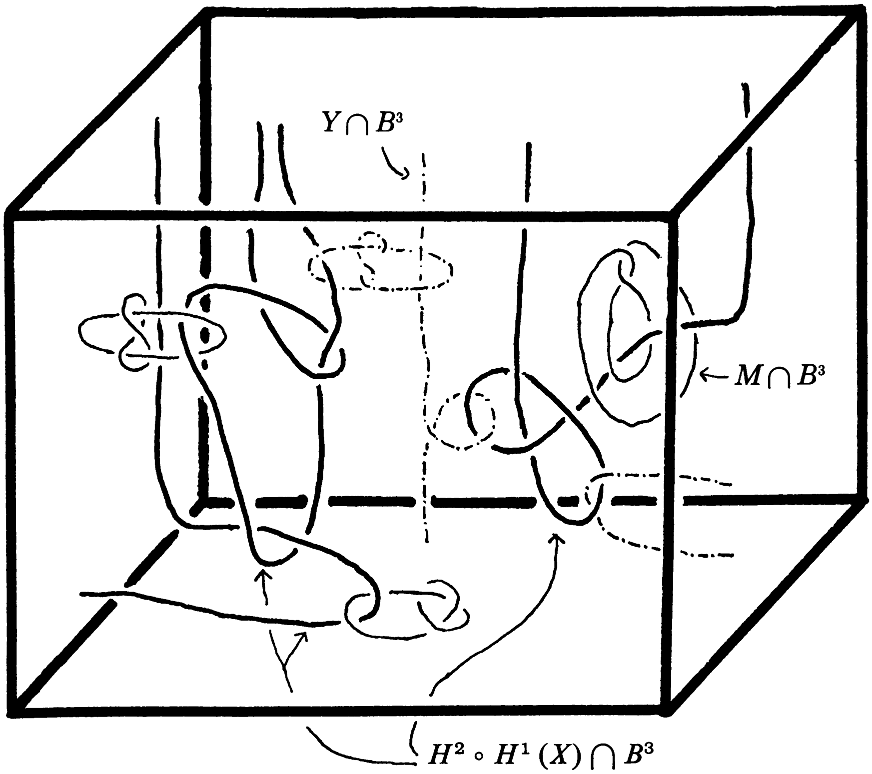

FIGURE 6

Step 4. There is a push h_t^4 on $\tilde{B}^4$ such that

(17) $h_t^4 | M \cup Y = \text{id}$,

(18) $\text{pr} | H^4 \circ H^3 \circ H^2 \circ H^1(X) \cap B^3$ is a homeomorphism, and

(19) each component of $H^4 \circ H^3 \circ H^2 \circ H^1(X) \cap B^3$ is an arc which spans B^3 from $T(B^3)$ to $B(B^3)$.

Figure 8 illustrates the action of H^4 on one component of $H^3 \circ H^2 \circ H^1(X) \cap B^3$ which has both endpoints in $T(B^3)$.

Let $B^2 = K(\{0\} \times [0, 1] \times [-1, 1] \times \{0\})$. See Figure 2. By (18), there exists a homeomorphism L from B^3 onto itself such that $L(T(B^3)) = T(B^3)$, $L(B(B^3)) = B(B^3)$, and $L(H^4 \circ H^3 \circ H^2 \circ H^1(X) \cap B^3) \subset B^2$. See Figure 9. At this point it becomes awkward to carry through the picture of B^3 given in sequence in Figures 3 through 8. Hence Figure 9 does not correspond to the preceding figures. Let $g_t^1 = (h_t^4)^{-1} * (h_t^3)^{-1} * (h_t^2)^{-1} * (h_t^1)^{-1}$. Then we have arrived at the following situation:

(20) for all t, $g_t^1(S_2 \cup \cdots \cup S_m) \cap (X \cup Y) = \varnothing$, and

(21) each component of $X \cap G^1(B^3)$ is an arc in $G^1 \circ L^{-1}(B^2)$ which spans $G^1(B^3)$ from $G^1(T(B^3))$ to $G^1(B(B^3))$.

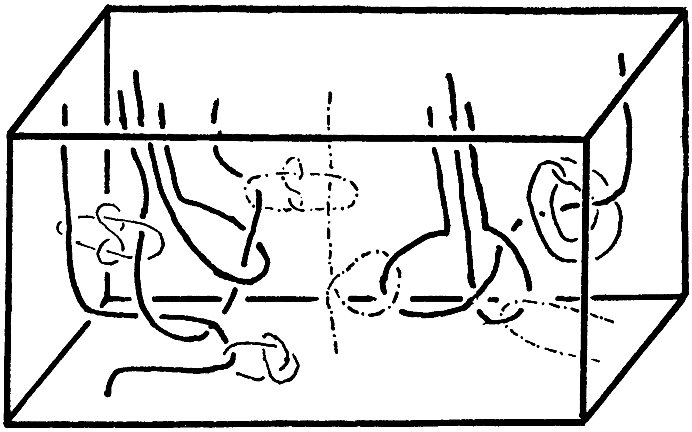

FIGURE 7a

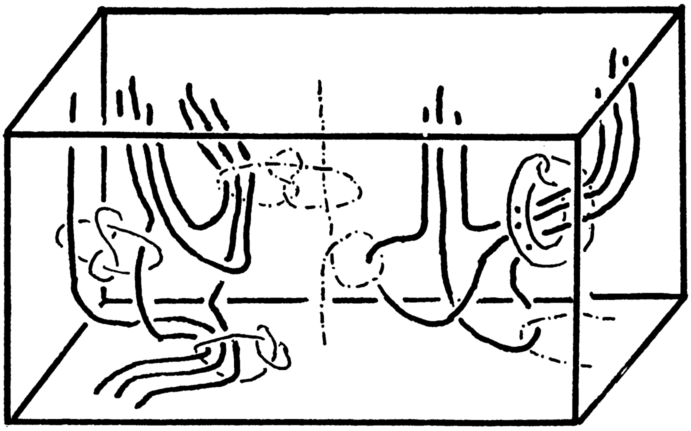

FIGURE 7b

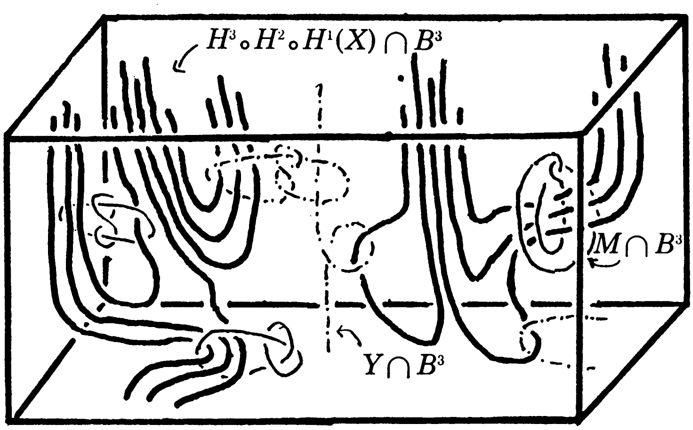

FIGURE 7c

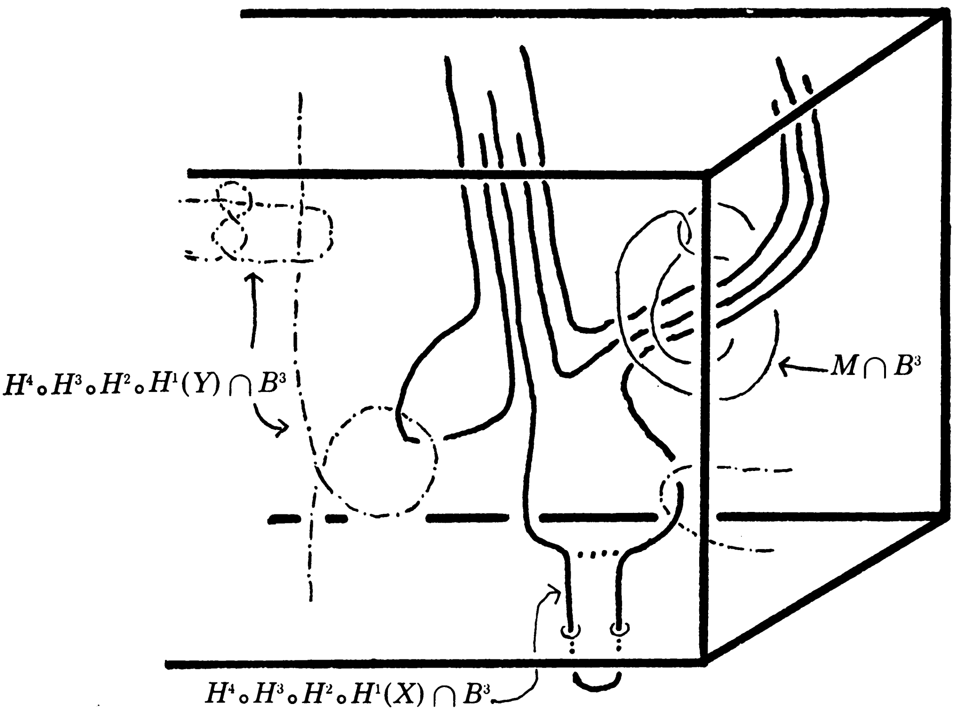

FIGURE 8

We may assume, using general position, that each component of $G^1(S_2 \cup \cdots \cup S_m) \cap G^1 \circ L^{-1}(B^2)$ is a singleton. Hence,

Step 5. There is a push g_t^2 on $G^1(\tilde{B}^4)$ such that

(22) $g_t^2 | G^1(B^3)$ is a push on $G^1(B^3)$ and

(23) $G^2 \circ G^1(S_2 \cup \cdots \cup S_m) \cap [G^1 \circ L^{-1}(B^2) \cup Y] = \varnothing$.

$g_t^2 | G^1(B^3)$ may be constructed as follows: If

$$G^1(S_2 \cup \cdots \cup S_m) \cap G^1 \circ L^{-1}(B^2) = \{y_1, \ldots, y_k\},$$

draw mutually disjoint arcs $A_1, \ldots,$ and A_k in $G^1 \circ L^{-1}(B^2)$ such that for each i, Int $A_i \subset$ Int $G^1 \circ L^{-1}(B^2)$, $A_i \cap Y = \varnothing$, y_i is an endpoint of A_i and the other endpoint is in Bd $(G^1 \circ L^{-1}(B^2)) \cap$ Int $G^1(B^3)$. Then, by a push g_t^2 which is the identity outside a neighborhood of $\bigcup \{A_i : 1 \leq i \leq k\}$ which does not intersect Y, $\{y_1, \ldots, y_k\}$ may be pushed off $G^1 \circ L^{-1}(B^2)$. See Figure 10.

Step 6. There is a push g_t^3 on $G^2 \circ G^1(\tilde{B}^4)$ such that

(24) $g_t^3 | G^2 \circ G^1(S_2 \cup \cdots \cup S_m) = \text{id}$,

(25) $g_t^3(G^2 \circ G^1(B^3)) \subset G^2 \circ G^1(B^3)$ for all t, and

(26) $G^3 \circ G^2 \circ G^1(B^3) \cap G^2 \circ G^1(L^{-1}(B^2)) = \varnothing$.

g_t^3 may be constructed as a push which pulls $G^2 \circ G^1(B^3)$ into itself and which is the identity outside a small neighborhood of $G^2 \circ G^1(L^{-1}(B^2))$.

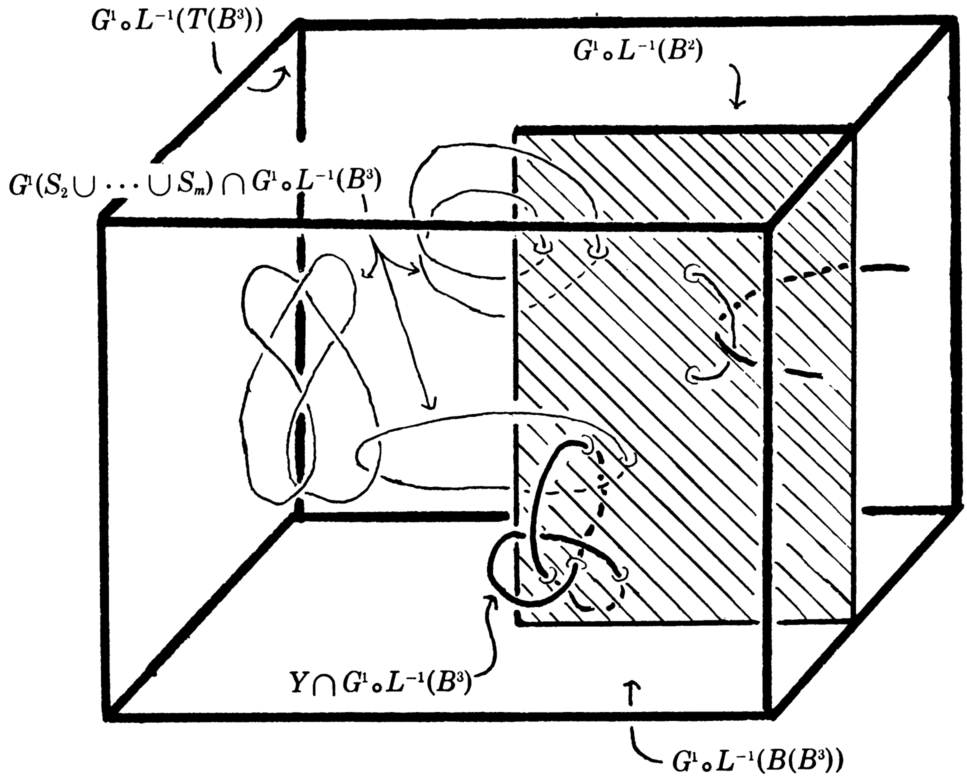

FIGURE 9

Let $h_t = g_t^3 * g_t^2 * g_t^1$. The conclusion of the lemma then follows from (21), (23), (24), and (26).

LEMMA 2. *Suppose $S_1, \ldots,$ and S_m are mutually disjoint polyhedral 2-spheres in* Int $(S^2 \times D^2)$ *which satisfy* (1) *and* (2) *of Lemma* 1, *and D_1 and D_2 are disks such that $S^2 = D_1 \cup D_2$. Suppose also, for $i=1$ or 2, $R(D_i)$ is a neighborhood of D_i on S^2 and $\tilde{D}^2$ is a polyhedral disk in* Int D^2. *Then there exists a push h_t on $S^2 \times D^2$ such that for each i there exists $j=1$ or 2 such that $h_1(S_i) \subset$* Int $(R(D_j) \times \tilde{D}^2)$.

Proof. We may assume $D_2 = \text{cl}\,(S^2 - D_1)$. (If not replace D_2 by $\text{cl}\,(S^2 - D_1)$.) Also, we may assume $S^2 - R(D_j) \neq \varnothing$ for $j=1$ or 2. (Otherwise the lemma is trivial.)

Choose $x_j \in S^2 - R(D_j)$. By Lemma 1, there is a push g_t on $S^2 \times D^2$ such that $g_1(S_1) \cap (\{x_1\} \times D^2) = \varnothing$ and $g_1(S_2 \cup \cdots \cup S_m) \cap (\{x_2\} \times D^2) = \varnothing$. For $j=1$ or 2 let N_j be a neighborhood of x_j on S^2 such that $(N_1 \times D^2) \cap g_1(S_1) = \varnothing$ and $(N_2 \times D^2) \cap g_1(S_2 \cup \cdots \cup S_m) = \varnothing$. There exists a push f_t on S^2 such that $f_1(S^2 - N_1) \subset R(D_1)$ and $f_1(S^2 - N_2) \subset R(D_2)$. Choose a disk C^2 in Int D^2 such that $g_1(S_1 \cup S_2 \cup \cdots \cup S_m) \subset S^2 \times C^2$. Extend f_t to a level preserving isotopy (still called f_t) on $S^2 \times C^2$. Then $f_1 \circ g_1(S_1) \subset R(D_1) \times C^2$ and $f_1 \circ g_1(S_2 \cup \cdots \cup S_m) \subset R(D_2) \times C^2$.

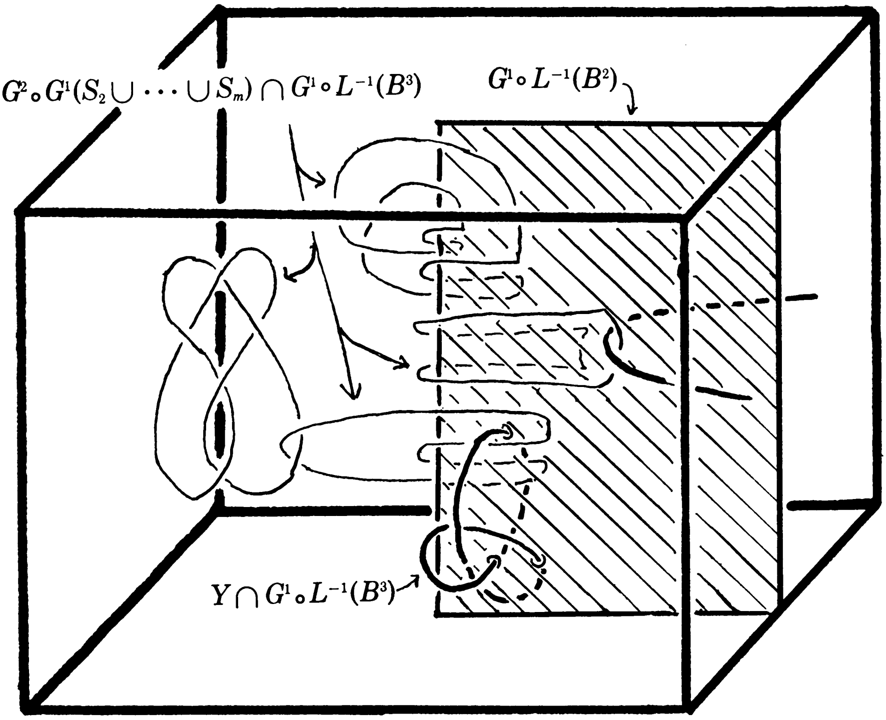

FIGURE 10

f_t may be extended to a push on $S^2 \times D^2$. Finally, there is a push k_t on $S^2 \times D^2$ such that, for each $x \in S^2$, $k_t(\{x\} \times D^2) = \{x\} \times D^2$ and

$$k_1 \circ f_1 \circ g_1(S_1 \cup S_2 \cup \cdots \cup S_m) \subset S^2 \times \tilde{D}^2.$$

Then let $h_t = k_t * f_t * g_t$.

LEMMA 3. *Suppose $S_1, \ldots,$ and S_m are mutually disjoint polyhedral 2-spheres in* Int $(S^2 \times D^2)$ *such that, for each i, there exists $j = 1$ or 2 such that*

$$S_i \subset \text{Int } R_e(Z^*((j-1)\pi, j\pi)),$$

where $e > 0$. Then there exists a push h_t on $S^2 \times D^2$ such that if

$$S_i \subset \text{Int } R_e(Z^*((j-1)\pi, j\pi))$$

then $S_i = D_i(0) \cup D_i(1)$ where, for $l = 0$ or 1, $h_1(D_i(l))$ is a disk in

$$\text{Int } R_e(W^*(l, l+1, (j-1)\pi, j\pi)).$$

At this point the notation becomes cumbersome and we shorten it to facilitate the proof. The notation described below is used only in the proof of Lemma 3.

Let

$$U(j) = Z^*((j-1)\pi, j\pi) \quad j = 1 \text{ or } 2,$$
$$U_l = A^*(l, l+1) \qquad l = 0 \text{ or } 1,$$

and

$$U_l(j) = U_l \cap U(j).$$

Note that each of these is a closed set. We may think of U_0 and U_1 as being the northern and southern hemispheres of the thickened globe. See Figure 11. Then

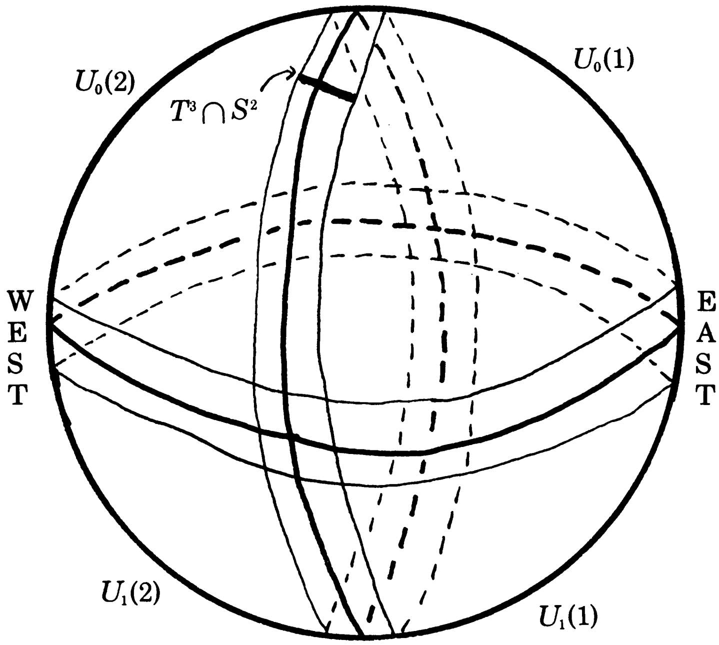

FIGURE 11

$U(1)$ and $U(2)$ are the eastern and western hemispheres and $U_l(j)$ denotes a quarter of the thickened globe. Thus the hypothesis says each S_i is contained in a neighborhood of either the eastern or western hemisphere. The conclusion says we may push the S_i's around so that, for each i, S_i is a union of disks, each of which is contained in a quarter sphere. We make one more simplification of notation. If $X = U_l$, $U(j)$, or $U_l(j)$ then we let $R(X) = R_e(X)$.

Proof of Lemma 3. Let $K^* = U_0 \cap U_1 = C^*(1)$. We assume each S_i is in general position relative to K^* and we assume also each S_i intersects K^*. (If a particular S_i does not intersect K^* it may be pushed so that it does intersect K^* and still is contained in one of $R(U(1))$ or $R(U(2))$. This may be done without moving any other S_j. The lack of this adjustment would create a necessity for special cases, both in the statement of the lemma and its proof.) Next, if necessary, we reorder $S_1, \ldots,$ and S_m so that there is an integer r such that $1 \leqq r \leqq m$, $S_i \subset R(U(1))$ if $1 \leqq i \leqq r$ and $S_i \subset R(U(2))$ if $r < i \leqq m$.

Step 1. There is a push f_t on $S^2 \times D^2$ such that

(1) $f_t = \text{id}$ on U_1 and

(2) for each i, S_i contains a disk-with-holes P_i where

(i) if $1 \leqq i \leqq r$ then $f_1(P_i) \subset R(U_1(1))$ and $\text{cl}\,(f_1(S_i - P_i)) \subset R(U_0(1))$ and

(ii) if $r < i \leqq m$ then $f_1(P_i) \subset R(U_1(2))$ and $\text{cl}\,(f_1(S_i - P_i)) \subset R(U_0(2))$.

f_t is constructed in two parts, denoted (A) and (B) below.

Now for each i, $S_i = Q_i(0) \cup Q_i(1)$ where $Q_i(0) = S_i \cap U_0$ and $Q_i(1) = S_i \cap U_1$. Then $Q_i(0) \cap Q_i(1) = \text{Bd}\, Q_i(0) = \text{Bd}\, Q_i(1)$ and each $Q_i(j)$ is a union of finitely many mutually disjoint disks-with-holes. Let $P_i(1), \ldots$, and $P_i(q_i)$ denote the components of $Q_i(1)$. For each i choose mutually disjoint arcs $A_i(1), \ldots$, and $A_i(q_i - 1)$ in $Q_i(0)$ such that if $1 \leqq l < q_i$ then $\text{Bd}\, A_i(l) \subset \text{Bd}\, Q_i(1)$ and $Q_i(1) \cup \{A_i(l) : 1 \leqq l \leqq q_i\}$ is connected. Let T^3 be the 3-cell $\{(\rho, \theta) : \theta = \pi/2 \text{ and } -e \leqq \rho \leqq e\} \times D^2$. See Figure 11. We may assume each $A_i(l)$ and each S_i is in general position relative to T^3. Then for each i and l, $A_i(l) \cap T^3$ is a finite set in $\text{Int}\, T^3$ and $(S_1 \cup \cdots \cup S_m) \cap T^3$ is a one dimensional polyhedron. See Figure 12.

Part (A). Push $\bigcup \{A_i(l) : 1 \leqq i \leqq m \text{ and } 1 \leqq l < q_i\}$ out of T^3. There exist standard cell pairs $(B^4(1), B^3(1))$ and $(B^4(2), B^3(2))$ such that

(3) $B^4(1) \cap B^4(2) = \varnothing$,

(4) for $k = 1$ or 2, $B^4(k) \subset \text{Int}\,[U_0(k) - R(U_1)]$,

(5) $B^4(1) \cap S_i = \varnothing$ if $r < i \leqq m$,

(6) $B^4(2) \cap S_i = \varnothing$ if $1 \leqq i \leqq r$,

(7) $B^3(k) - T^3 \neq \varnothing$ if $k = 1$ or 2,

(8) $B^4(k) \cap T^3 = B^3(k) \cap T^3$ for $k = 1$ or 2,

(9) $B^3(1) \cap T^3$ contains $T^3 \cap [\bigcup \{A_i(l) : 1 \leqq i \leqq r \text{ and } 1 \leqq l < q_i\}]$, and

(10) $B^3(2) \cap T^3$ contains $T^3 \cap [\bigcup \{A_i(l) : r < i \leqq m \text{ and } 1 \leqq l < q_i\}]$.

Then there exists a push φ_t^1 on $B^3(1)$ such that $\varphi_1^1(A_i(l) \cap T^3) \subset B^3(1) - T^3$ if $1 \leqq i \leqq r$ and $1 \leqq l < q_i$ and there exists a push φ_t^2 on $B^3(2)$ such that $\varphi_1^2(A_i(l) \cap T^3) \subset B^3(2) - T^3$ if $r < i \leqq m$ and $1 \leqq l < q_i$. Each φ_t^k may be extended to a push (still called φ_t^k) on $B^4(k)$. Then we define w_t to be the push on $S^2 \times D^2$ such that $w_t = \text{id}$ outside $B^4(1) \cup B^4(2)$ and $w_t | B^4(k) = \varphi_t^k$ for $k = 1$ or 2.

Part (B). Push $\bigcup \{w_1(A_i(l)) : 1 \leqq i \leqq m \text{ and } 1 \leqq l < q_i\}$ into $R(U_0) \cap R(U_1)$. Now

$$T^3 \cap [\bigcup \{w_1(A_i(l)) : 1 \leqq i \leqq m \text{ and } 1 \leqq l < q_i\}] = \varnothing,$$

hence there exists a 2-cell T^2 on S^2 such that $T^3 \cap S^2 \subset T^2$ and

$$(T^2 \times D^2) \cap [\bigcup \{w_1(A_i(l)) : 1 \leqq i \leqq m \text{ and } 1 \leqq l < q_i\}] = \varnothing.$$

Let $\tilde{D}^2$ be a disk in $\text{Int}\, D^2$ such that $S^2 \times \tilde{D}^2$ contains $\bigcup \{w_1(S_i) : 1 \leqq i \leqq m\}$. Then there exists a push v_t on S^2 such that

(11) $v_t(U_0(j) \cap S^2) = U_0(j) \cap S^2$ for $j = 1$ or 2,

(12) $v_t(R(U_0(j)) \cap S^2) = R(U_0(j) \cap S^2)$ for $j = 1$ or 2,

(13) $v_t = \text{id}$ on $U_1 \cap S^2$ and

(14) $\text{Int}\, v_1(T^2)$ contains $\text{cl}\,(U_0 - R(U_1)) \cap S^2$.

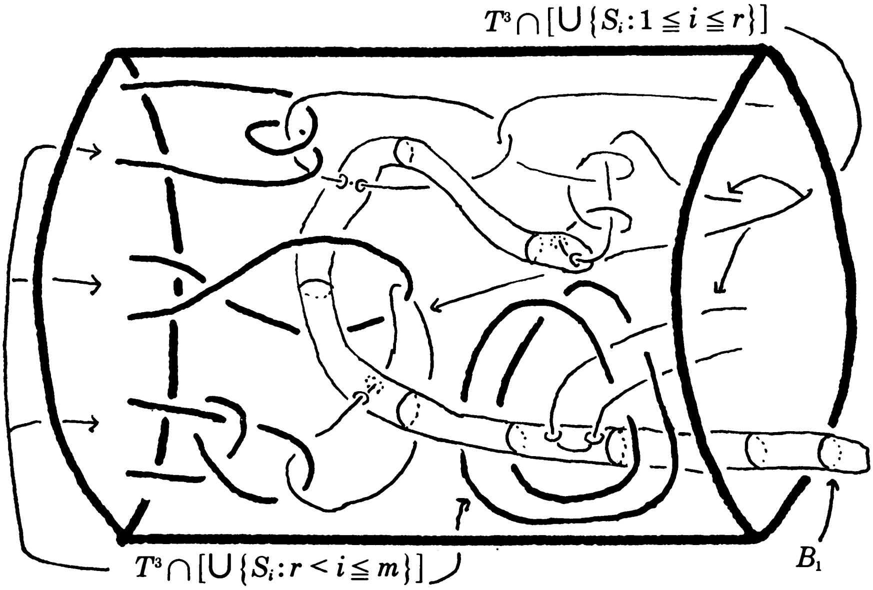

FIGURE 12

Then we extend v_t levelwise to an isotopy (not a push) on $S^2 \times \tilde{D}^2$ and then to a push (still called v_t) on $S^2 \times D^2$. Let $f_t = v_t * w_t$. Then

$$f_1(A_i(l)) \subset \text{Int}\,[R(U_0(1)) \cap R(U_1(1))] \quad \text{if } 1 \leqq i \leqq r,$$

and

$$f_1(A_i(l)) \subset \text{Int}\,[R(U_0(2)) \cap R(U_1(2))] \quad \text{if } r < i \leqq m.$$

Now $Q_i(1) \cup A_i(1) \cup \cdots \cup A_i(q_i-1)$ is a connected subset of S_i, hence there exists a disk-with-holes P_i such that $Q_i(1) \cup A_i(1) \cup \cdots \cup A_i(q_i-1) \subset P_i \subset S_i$,

(15) $f_1(P_i) \subset \text{Int}\, R(U_1(1))$ if $1 \leqq i \leqq r$, and

(16) $f_1(P_i) \subset \text{Int}\, R(U_1(2))$ if $r < i \leqq m$.

Hence Step 1 is accomplished.

Then for each i, $S_i = P_i \cup E_i(1) \cup \cdots \cup E_i(z_i)$ where P_i is a disk-with-holes, $E_i(1), \ldots,$ and $E_i(z_i)$ are the components of $S_i - \text{Int}\, P_i$,

(17) if $1 \leqq i \leqq r$ then $f_1(P_i) \subset R(U_1(1))$ and $f_1(E_i(1) \cup \cdots \cup E_i(z_i)) \subset R(U_0(1))$, and

(18) if $r < i \leqq m$ then $f_1(P_i) \subset R(U_1(2))$ and $f_1(E_i(1) \cup \cdots \cup E_i(z_i)) \subset R(U_0(2))$.

Now for each i construct mutually disjoint arcs $C_i(1), \ldots,$ and $C_i(z_i-1)$ in P_i such that, for each l, $\text{Int}\, C_i(l) \subset \text{Int}\, P_i$ and $C_i(l)$ has one endpoint on Bd $E_i(l)$ and the other on Bd $E_i(l+1)$.

Step 2. Using the techniques of Step 1 we may find a push g_t on $S^2 \times D^2$ such that

(19) $g_t = \text{id}$ outside U_1,

(20) if $S_i \subset \text{Int}\, R(U(j))$ then $g_1 \circ f_1(S_i) \subset \text{Int}\, R(U(j))$,

(21) $g_1 \circ f_1(C_i(1) \cup \cdots \cup C_i(z_i-1)) \subset \text{Int}\,[R(U_0(1)) \cap R(U_1(1))]$ if $1 \leqq i \leqq r$, and

(22) $g_1 \circ f_1(C_i(1) \cup \cdots \cup C_i(z_i-1)) \subset \text{Int}\,[R(U_0(2)) \cap R(U_1(2))]$ if $r < i \leqq m$.

Since $E_i(1) \cup \cdots \cup E_i(z_i) \cup C_i(1) \cup \cdots \cup C_i(z_i-1)$ is a connected and simply connected subset of S_i, there exists a disk $D_i(0)$ such that

$$E_i(1) \cup \cdots \cup E_i(z_i) \cup C_i(1) \cup \cdots \cup C_i(z_i-1) \subset D_i(0),\ D_i(0) \subset S_i,$$

(23) $g_1 \circ f_1(D_i(0)) \subset \text{Int } R(U_0(1))$ if $1 \leqq i \leqq r$, and
(24) $g_1 \circ f_1(D_i(0)) \subset \text{Int } R(U_0(2))$ if $r < i \leqq m$.
Then $D_i(1) = S_i - \text{Int } D_i(0)$ is a disk in P_i. Hence
(25) $g_1 \circ f_1(D_i(1)) \subset \text{Int } R(U_1(1))$ if $1 \leqq i \leqq r$ and
(26) $g_1 \circ f_1(D_i(1)) \subset \text{Int } R(U_1(2))$ if $r < i \leqq m$.
Then the proof is finished if we let $h_t = g_t * f_t$.

Proof of Proposition (1, 1). In this case we have $0 = t_0 < t_1 < t_2 = 2$ and $0 = a_1 < a_2 < a_3 = 2\pi$. Now $Z(a_0, a_1)$ and $Z(a_1, a_2)$ are disks and

$$S^2 = Z(a_0, a_1) \cup Z(a_1, a_2).$$

Also, $F(S_1), \ldots,$ and $F(S_{m(0)})$ are polyhedral 2-spheres in Int $(S^2 \times D^2)$ and, since the decomposition is simple, we may assume (1) and (2) of Lemma 1 are satisfied. Hence, by Lemma 2, there is a push v_t on $S^2 \times D^2$ such that if $1 \leqq i \leqq m(0)$ then $v_1(F(S_i)) \subset \text{Int } R_e(Z^*(a_{j-1}, a_j))$ for $j = 1$ or 2. Then $v_1(F(S_1)), \ldots,$ and $v_1(F(S_{m(0)}))$ satisfy the hypothesis of Lemma 3. Therefore there is a push w_t on $S^2 \times D^2$ such that for each $i = 1, \ldots,$ or $m(0)$ there exists $j = 1$ or 2 such that $S_i = D_i(0) \cup D_i(1)$ where $D_i(0)$ and $D_i(1)$ are disks and for $l = 0$ or 1,

$$w_1 \circ v_1(F(D_i(l))) \subset \text{Int } R_e(W^*(t_l, t_{l+1}, a_{j-1}, a_j)).$$

For each i and l choose a neighborhood $R(D_i(l))$ of $D_i(l)$ on S_i such that

$$w_1 \circ v_1 \circ F(R(D_i(l))) \subset \text{Int } R_e(W^*(t_l, t_{l+1}, a_{j-1}, a_j)).$$

For each $i = 1, \ldots,$ or $m(0)$ choose a disk $\tilde{D}_i$ in Int D_i such that if

$$w_1 \circ v_1 \circ F(D_i(l)) \subset \text{Int } R_e(W^*(t_l, t_{l+1}, a_{j-1}, a_j))$$

then $w_1 \circ v_1 \circ F(R(D_i(l)) \times \tilde{D}_i) \subset \text{Int } R_e(W^*(t_l, t_{l+1}, a_{j-1}, a_j))$.

By Lemma 2, for each $i = 1, \ldots,$ or $m(0)$, there is a push g_t^i on $w_1 \circ v_1(F(S_i \times D_i))$ such that if $i' = 1, \ldots,$ or m_i then $g_1^i \circ w_1 \circ v_1 \circ F(S_{ii'})$ is contained in one of $w_1 \circ v_1 \circ F(R(D_i(0)) \times \tilde{D}_i)$ or $w_1 \circ v_1 \circ F(R(D_i(1)) \times \tilde{D}_i)$, and hence in

$$\text{Int } R_e(W^*(t_l, t_{l+1}, a_{j-1}, a_j)) \quad \text{for some } l \text{ and } j.$$

Let u_t be the push on $S^2 \times D^2$ such that $u_t | w_1 \circ v_1 \circ F(S_i \times D_i) = g_t^i$ and $u_t = \text{id}$ elsewhere. Finally, if α is a stage 2 index there exists a push f_t^α on $u_1 \circ w_1 \circ v_1 \circ F(X_\alpha)$ such that $f_1^\alpha \circ u_1 \circ w_1 \circ v_1 \circ F(X_\alpha \cap M_3) \subset \text{Int } R_e(W^*(t_l, t_{l+1}, a_{j-1}, a_j))$. Let

$$k_t | u_1 \circ w_1 \circ v_1 \circ F(X_\alpha) = f_t^\alpha \quad \text{and} \quad k_t = \text{id elsewhere}.$$

Finally, let $h_t = k_t * u_t * w_t * v_t$.

5. **Proposition** $(r, 1)$.

LEMMA 4. *Suppose* $0=s_0<s_1<\cdots<s_{r-1}<s_r=2$ *and* $0=a_0<a_1<a_2=2\pi$, *and* $e>0$. *Suppose* $S_1,\ldots,$ *and* S_m *are mutually disjoint polyhedral 2-spheres in* $\text{Int}(S^2\times D^2)$ *such that for each i, there exists* $j=1,\ldots,$ *or r and* $k=1$ *or* 2 *such that*

$$S_i\subset \text{Int } R_e(W^*(s_{j-1}, s_j, a_{k-1}, a_k)).$$

Suppose also $t_0,\ldots,$ *and* t_{r+1} *are numbers such that* $t_0=s_0$, $t_{r+1}=s_r$ *and for* $i=1,\ldots,$ *or r*, $s_{i-1}<t_i<s_i$. *Then there exists a push* h_t *on* $S^2\times D^2$ *such that for each* $i=1,\ldots,$ *or m*, $S_i=E_i(0)\cup E_i(1)$ *where, for* $l=1$ *or* 2, $E_i(l)$ *is a disk and there exists* $j=1,\ldots,$ *or* $r+1$ *and* $k=1$ *or* 2 *such that*

$$h_1(E_i(l))\subset \text{Int } R_e(W(t_{j-1}, t_j, a_{k-1}, a_k)).$$

Proof. Again we shorten the notation to facilitate the proof. The notation described below is used only in the proof of Lemma 4.

Let $V_j(k)=W^*(s_{j-1}, s_j, a_{k-1}, a_k)$, $V_j'(k)=W^*(s_{j-1}, t_j, a_{k-1}, a_k)$, and

$$V_j''(k) = W^*(t_j, s_j, a_{k-1}, a_k)$$

for $j=1,\ldots,$ or r and $k=1$ or 2. See Figure 13.

For $j=1,\ldots,$ or r we let $V_j=V_j(1)\cup V_j(2)$, $V_j'=V_j'(1)\cup V_j'(2)$, and $V_j''=V_j''(1)\cup V_j''(2)$. If X is any of the sets defined above we let $R(X)=R_e(X)$. We shall assume e is small enough so that if X and Y are any two of the sets defined above and $X\cap Y=\varnothing$ then

(1) $R(X)\cap R(Y)=\varnothing$.

Now we choose an indexing function I from $\{S_1,\ldots,S_m\}$ into

$$\{(j,k);\ 1\leqq j\leqq r \text{ and } k=1 \text{ or } 2\}$$

such that $I(S_i)=(j,k)$ implies $S_i\subset\text{Int } R(V_j(k))$. We shall assume, to avoid special cases, that $I(S_i)=(j,k)$ implies $S_i\cap C^*(t_j)=S_i\cap(V_j'\cap V_j'')\neq\varnothing$ and that each S_i is in general position with respect to each $C^*(t_j)$.

Step 1. *Adjustments in* V_1 *and* V_r. In this step we construct a push w_t on $S^2\times D^2$ such that

(2) $w_t=\text{id}$ outside $V_1'\cup V_r''$,

(3) if $I(S_i)=(j,k)$ then $w_1(S_i)\subset\text{Int } R(V_j(k))$, and

(4) if $I(S_i)=(1,k)$ or (r,k) then $S_i=P_i\cup D_i(1)\cup\cdots\cup D_i(q_i)$ where P_i is a disk-with-holes, $D_i(1),\ldots,$ and $D_i(q_i)$ are the components of $S_i-\text{Int } P_i$ and

(a) if $I(S_i)=(1,k)$ then $w_1(P_i)\subset\text{Int } R(V_1''(k))$ and, for each l,

$$w_1(D_i(l))\subset \text{Int } R(V_1'(k)),$$

and

(b) if $I(S_i)=(r,k)$ then $w_1(P_i)\subset\text{Int } R(V_r'(k))$ and, for each l,

$$w_1(D_i(l))\subset \text{Int } R(V_r''(k)).$$

The construction of w_t in V_1 is done exactly as in Step 1 of Lemma 3. (In that proof replace U_0 by V_1' and U_1 by $V_1'' \cup V_2 \cup \cdots \cup V_r$.) $w_t | V_r$ is constructed in a similar manner. See Figure 13. Figures 13 through 15 should not be considered to be pictures of the situation but to be schematic diagrams to help visualize the pushes.

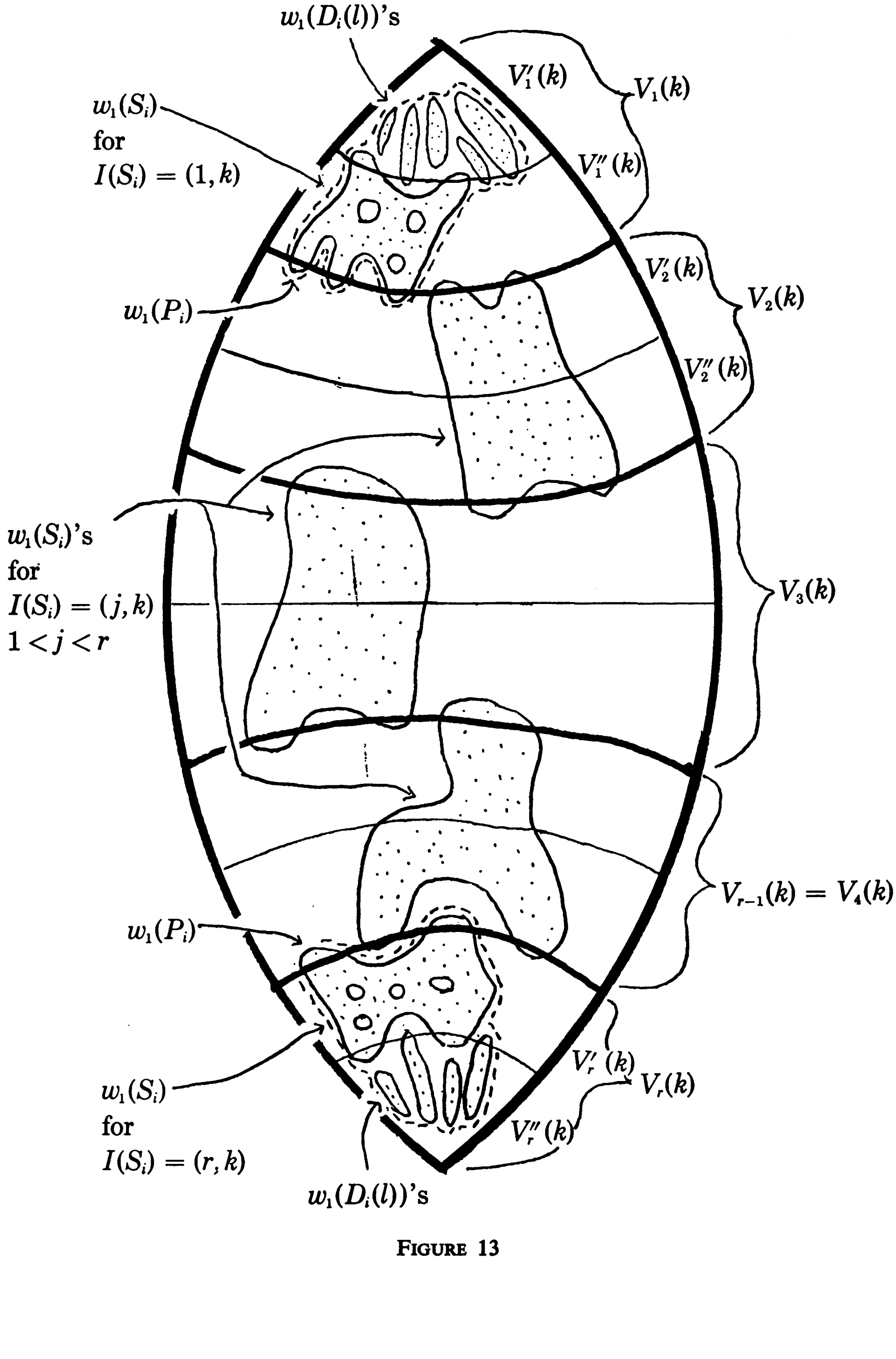

FIGURE 13

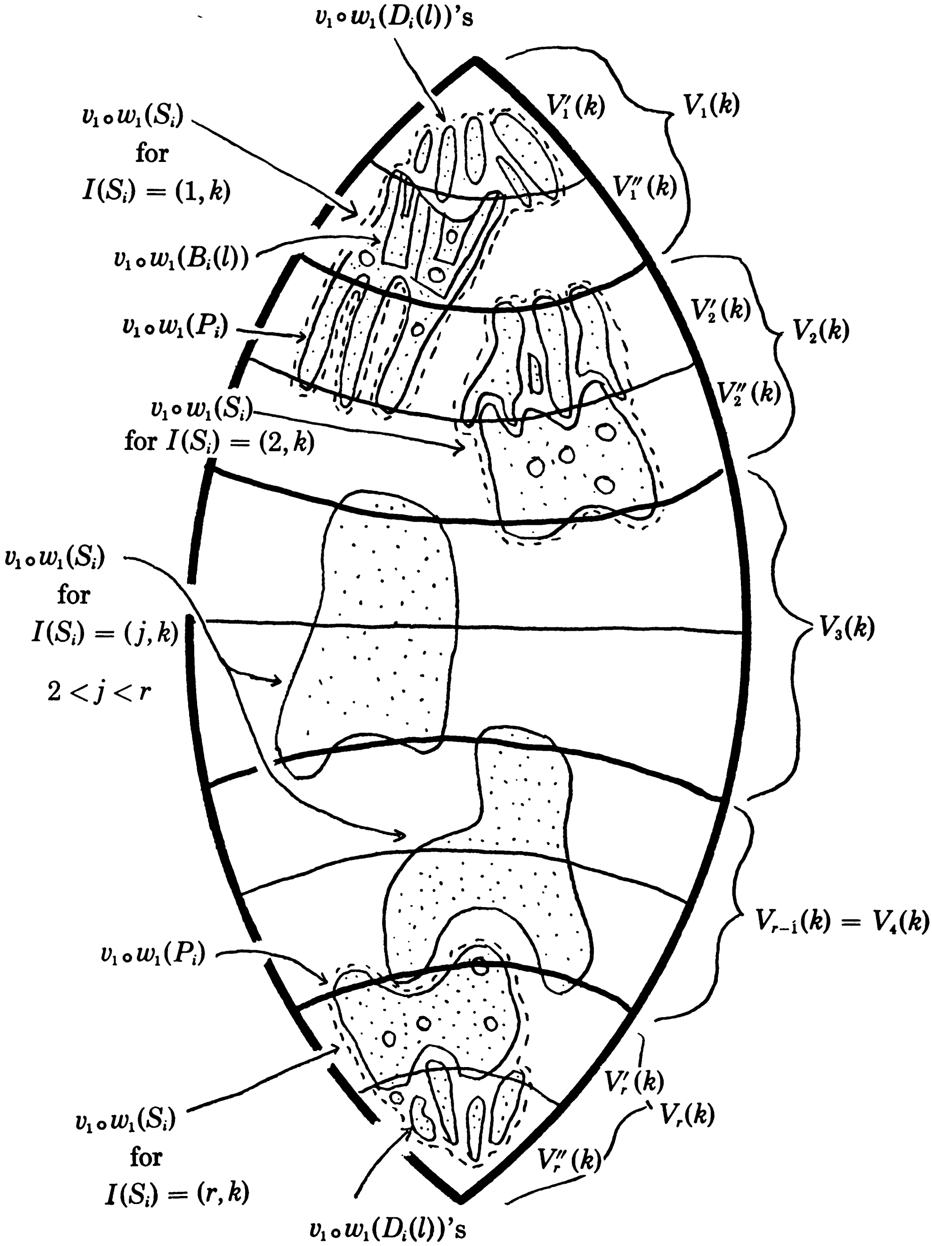

FIGURE 14

Now if $I(S_i) = (1, k)$, choose arcs $B_i(1), \ldots,$ and $B_i(q_i - 1)$ in P_i such that, for each l, Int $B_i(l) \subset$ Int P_i and $B_i(l)$ has one endpoint on Bd $D_i(l)$ and the other on Bd $D_i(l+1)$.

Step 2. *Adjustments in* V_2. In this step we construct a push v_t on $S^2 \times D^2$ such that

(5) $v_t = \text{id}$ outside $R(V_2')$,

(6) if $I(S_i)=(j, k)$ where $j>2$ then $v_1 \circ w_1(S_i)=w_1(S_i) \subset \text{Int } R(V_j(k))$,

(7) if $I(S_i)=(1, k)$ then $v_1 \circ w_1(D_i(1) \cup \cdots \cup D_i(q_i)) \subset \text{Int } R(V_1'(k))$,

(8) if $I(S_i)=(1, k)$ then $v_1 \circ w_1(B_i(1) \cup \cdots \cup B_i(q_i-1)) \subset \text{Int } R(V_1''(k))$,

(9) if $I(S_i)=(1, k)$ then $v_1 \circ w_1(P_i) \subset \text{Int } [R(V_1''(k)) \cup R(V_2'(k))]$, and

(10) if $I(S_i)=(2, k)$ then $S_i=P_i \cup D_i(1) \cup \cdots \cup D_i(q_i)$ where P_i is a disk-with-holes, $D_i(1), \ldots$, and $D_i(q_i)$ are the components of

$$S_i - \text{Int } P_i,\ v_1 \circ w_1(P_i) \subset \text{Int } R(V_2''(k))$$

and, for each l, $v_1 \circ w_1(D_i(l)) \subset \text{Int } R(V_2'(k))$.

Figure 14 illustrates the result of applying v_1.

Let H_1 be a projection along meridians of

$$R(V_2') = A^*(s_1-e, t_2+e) = A(s_1-e, t_2+e) \times D^2$$

onto $C^*(t_2)=C(t_2) \times D^2$. That is, if $((\rho, \theta), d) \in R(V_2')$ where $(\rho, \theta) \in A(s_1-e, t_2+e)$ and $d \in D^2$ then

$$H_1((\rho, \theta), d) = ((s_2, \theta), d).$$

Then H_1 is the final stage of a pseudo-isotopy H_t such that $H_0 = \text{id}$ where

$$H_t((\rho, \theta), d) = (((s_2-\rho)t+\rho, \theta), d).$$

Note that

(11) For each t and k, $H_t(R(V_2(k))) \subset R(V_2(k))$, $H_t(R(V_2'(k))) \subset R(V_2'(k))$ and $H_t(R(V_2''(k))) \subset R(V_2''(k))$.

Now if $I(S_i)=(2, k)$, let $w_1(S_i)=S_i=Q_i' \cup Q_i''$ where $Q_i'=S_i \cap (V_1'' \cup V_2')$ and $Q_i''=S_i \cap (V_2'' \cup V_3')$. By the general position, each of Q_i' and Q_i'' is a union of finitely many mutually disjoint disks-with-holes and $\text{Bd } Q_i' = \text{Bd } Q_i'' = S_i \cap C^*(t_2)$. Let the components of Q_i'' be $C_i(1), \ldots$, and $C_i(z_i)$. Choose mutually disjoint arcs $A_i(1), \ldots$, and $A_i(z_i-1)$ in Q_i' such that, for each $l=1, \ldots$, or z_i-1, $\text{Bd } A_i(l) \subset \text{Bd } Q_i''$ and $Q_i' \cup [\bigcup \{A_i(l) : 1 \leqq l \leqq z_i\}]$ is connected. Hence for each l, $A_i(l) \subset V_1'' \cup V_2'$. We make a general position adjustment of each S_i so that

$$H_1 \bigm| \bigcup \{A_i(l) : I(S_i) = (2, k) \text{ and } 1 \leqq l < z_i\}$$

is a homeomorphism. In fact, we may assume that if $I(S_i)=(2, k)$ and $1 \leqq l < z_i$ then $B_i^2(l) = \bigcup \{H_t(A_i(l)) : 0 \leqq t \leqq 1\}$ is an unknotted 2-cell and $B_i^2(l) \cap B_j^2(k) = \varnothing$ if $i \neq j$ or $l \neq k$. By another general position adjustment we may assume

$$\bigcup \{B_i^2(l) : I(S_i) = (2, k) \text{ and } 1 \leqq l < z_i\}$$

is disjoint from

$$\bigcup \{w_1(B_i(l)) : I(S_i) = (1, k) \text{ and } 1 \leqq l < q_i\}.$$

For each i and l choose a 4-cell $B_i^4(l)$ such that

(12) $B_i^4(l) \subset \text{Int } R(V_2'(l))$,

(13) $B_i^2(l) \subset \text{Int } B_i^4(l)$,

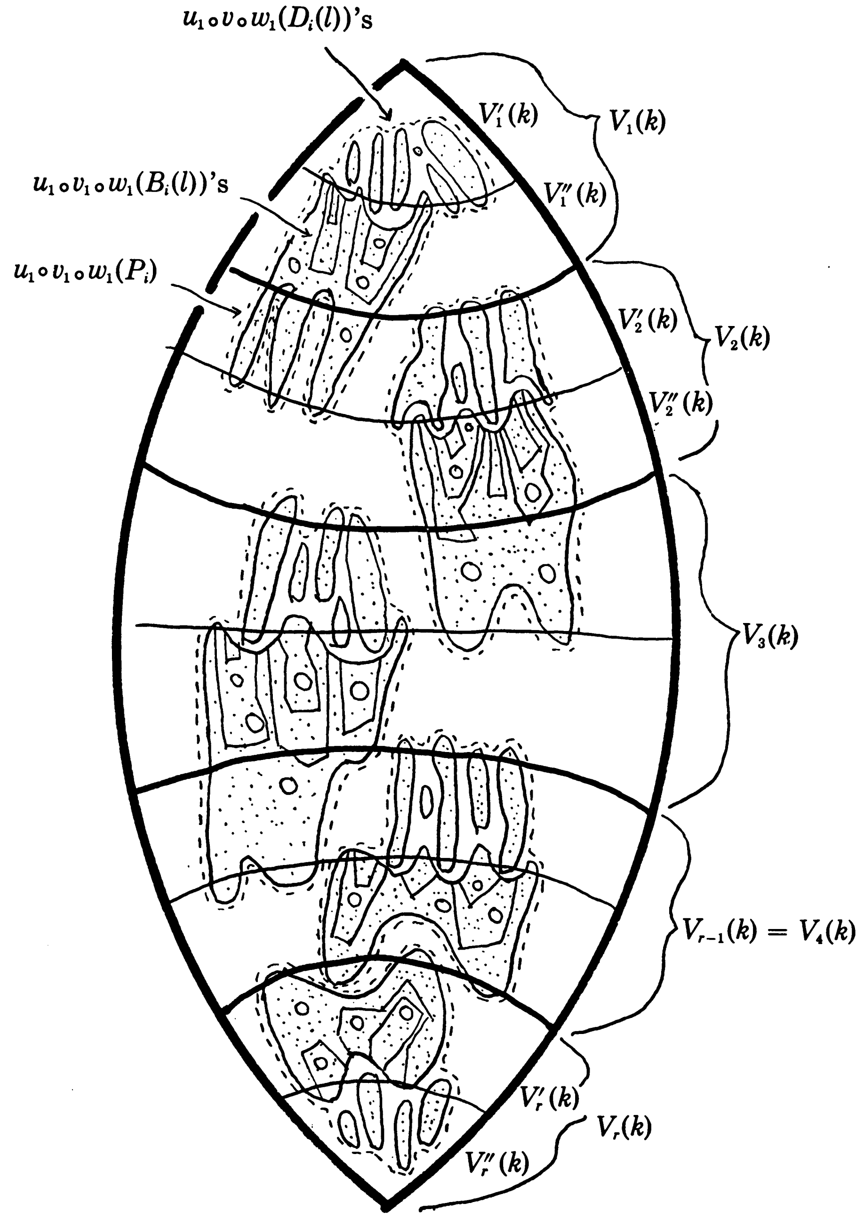

FIGURE 15

(14) $B_i^4(l) \cap B_j^4(k) = \varnothing$ if $i \neq j$ or $l \neq k$,

(15) $B_i^4(l)$ is disjoint from

$$\bigcup \{w_1(B_i(l)) : I(S_i) = (1, k) \text{ and } 1 \leqq l < q_i\},$$

and

(16) if $((\rho, \theta), d) \in B_i^4(l)$ then $H_t((\rho, \theta), d) \in B_i^4(l)$ for all t.

We now describe v_t by constructing it as a push on each $B_i^4(l)$. Fix integers i and l such that $I(S_i)=(2, k)$ and $1 \leqq l < z_i$. Then there is a push v_t on $B_i^4(l)$ such that

(17) $v_1(A_i(l)) \subset \text{Int}\,[R(V_2'(k)) \cap R(V_2''(k))]$ and

(18) if $x \in B_i^4(l)$ and $0 \leqq t \leqq 1$ then there exists t' such that $v_t(x)=H_{t'}(x)$.

Then v_t is extended outside $\bigcup \{B_i^4(l)\}$ by the identity. Then (5) follows from (12) and (12) also implies each $B_i^4(l)$ is disjoint from $R(V_j)$ if $j>2$, hence (6) is satisfied. Now if $I(S_i)=(1, k)$ and $1 \leqq l \leqq q_i$ then $w_1(D_i(l)) \subset \text{Int}\, R(V_1'(k))$. But (12) implies each $B_j^4(l')$ is disjoint from $R(V_1'(k))$, hence $v_1 \circ w_1(D_i(l))=w_1(D_i(l))$ and (7) is satisfied. Similarly, (15) implies (8) and (4a), (11) and (18) imply (9).

Suppose $I(S_i)=(2, k)$. Recall $w_1(S_i)=Q_i' \cup Q_i''$ where $Q_i' \subset \text{Int}\, R(V_2'(k))$ and $Q_i'' \subset \text{Int}\, R(V_2''(k))$. (11), (17) and (18) imply $v_1(Q_i') \subset \text{Int}\, R(V_2'(k))$ and

$$v_1(Q_i'' \cup A_i(1) \cup \cdots \cup A_i(z_i-1)) \subset \text{Int}\, R(V_2''(k)).$$

Now $(w_1)^{-1}(Q_i'' \cup A_i(1) \cup \cdots \cup A_i(z_i-1))$ is a connected subset of S_i, therefore there exists a disk-with-holes P_i on S_i such that

$$(w_1)^{-1}(Q_i'' \cup A_i(1) \cup \cdots \cup A_i(z_i-1)) \subset P_i$$

and

$$v_1 \circ w_1(P_i) \subset \text{Int}\, R(V_2''(k)).$$

Then if $D_i(1), \ldots,$ and $D_i(q_i)$ denote the components of $S_i - \text{Int}\, P_i$,

$$v_1 \circ w_1(D_i(1) \cup \cdots \cup D_i(q_i)) \subset v_1(Q_i')$$

and $v_1(Q_i') \subset \text{Int}\, R(V_2'(k))$. Hence (10) is satisfied.

Step 3. *Adjustments in* $V_3, V_4, \ldots,$ *and* V_{r-1}. In this step we construct a push u_t on $S^2 \times D^2$ such that

(19) $u_t = \text{id}$ outside $R(V_3') \cup R(V_4') \cup \cdots \cup R(V_{r-1}')$,

(20) if $I(S_i)=(j, k)$ where $1 \leqq j \leqq r-1$ then $S_i = P_i \cup D_i(1) \cup \cdots \cup D_i(q_i)$ where P_i is a disk-with-holes, $D_i(1), \ldots, D_i(q_i)$ are the components of $S_i - \text{Int}\, P_i$, P_i contains arcs $B_i(1), \ldots,$ and $B_i(q_i-1)$ such that if $1 \leqq l < q_i$ then $B_i(l)$ has one end-point on Bd $D_i(l)$ and the other on $D_i(l+1)$ and Int $B_i(l) \subset \text{Int}\, P_i$,

$$u_1 \circ v_1 \circ w_1(D_i(1) \cup \cdots \cup D_i(q_i)) \subset \text{Int}\, R(V_j'(k)),$$

$$u_1 \circ v_1 \circ w_1(B_i(1) \cup \cdots \cup B_i(q_i-1)) \subset \text{Int}\, R(V_j''(k)),$$

and

(a) if $j < r-1$ then $u_1 \circ v_1 \circ w_1(P_i) \subset \text{Int}\,[R(V_j''(k)) \cup R(V_{j+1}'(k))]$, and

(b) if $j = r-1$ then $u_1 \circ v_1 \circ w_1(P_i) \subset \text{Int}\, R(V_j''(k))$ (the special case, (b), arises because no adjustment is made in V_r in this step), and

(21) if $I(S_i)=(r, k)$ then $S_i = P_i \cup D_i(1) \cup \cdots \cup D_i(q_i)$ where P_i is a disk-with-holes, $D_i(1), \ldots,$ and $D_i(q_i)$ are the components of

$$S_i - \text{Int}\, P_i,\ u_1 \circ v_1 \circ w_1(P_i) \subset \text{Int}\, R(V_r'(k))$$

and

$$u_1 \circ v_1 \circ w_1(D_i(1) \cup \cdots \cup D_i(q_i)) \subset \text{Int}\, R(V_r''(k)).$$

Figure 15 illustrates the result of u_1.

Now (21) will follow from (4b) since $v_t|V_r=u_t|V_r=\text{id}$ for all t. We construct u_t as follows. Let $u_t=\text{id}$ on $R(V_2')\cup R(V_1)\cup R(V_r)$. Then for $j=1$, (20) follows from (7), (8), and (9). Now for $j=2$, from (10), $S_i=P_i\cup D_i(1)\cup\cdots\cup D_i(q_i)$ where P_i is a disk-with-holes, etc. We choose arcs $B_i(1),\ldots,$ and $B_i(q_i-1)$ in P_i such that if $1\leqq l<q_i$ then $B_i(l)$ has one endpoint on Bd $D_i(l)$ and the other on Bd $D_i(l+1)$ and Int $B_i(l)\subset$ Int P_i. Now we construct a push g_t^3 on $S^2\times D^2$ such that $g_t^3=\text{id}$ outside $R(V_3')$ so that (20) is satisfied for $j=2$ with g_t^3 replacing u_t and such that if $I(S_i)=(3,k)$ then $S_i=P_i\cup D_i(1)\cup\cdots\cup D_i(q_i)$ where P_i is a disk-with-holes, $D_i(1),\ldots,$ and $D_i(q_i)$ are the components of $S_i-\text{Int } P_i$,

$$g_1^3\circ v_1\circ w_1(P_i)\subset \text{Int } R(V_3''(k))$$

and

$$g_1^3\circ v_1\circ w_1(D_i(1)\cup\cdots\cup D_i(q_i))\subset \text{Int } R(V_3'(k)).$$

g_t^3 is constructed exactly as v_t is constructed in Step 2. Next we choose $B_i(1),\ldots,$ and $B_i(q_i-1)$ for all i such that $I(S_i)=(3,k)$ and, using the procedure of Step 2 again, construct a push g_t^4 on $S^2\times D^2$ such that $g_t^4=\text{id}$ outside $R(V_4')$ and such that (20) is satisfied with g_t^4 replacing u_t. In this manner we construct pushes $g_t^5,\ldots,$ and g_t^r such that if $5\leqq k\leqq r$ then $g_t^k=\text{id}$ outside $R(V_k')$ and so that is $j=k$ then (20) is satisfied if u_t is replaced by g_t^k. Then we let $u_t|R(V_k')=g_t^k|R(V_k')$ for $k=3,\ldots,$ or $r-1$ and $u_t=\text{id}$ elsewhere.

Step 4. *A second adjustment in* V_r. In this step we construct a push k_t on $S^2\times D^2$ such that

(22) $k_t=\text{id}$ outside $R(V_r'(k))$,

(23) if $I(S_i)=(r,k)$ then $S_i=E_i(0)\cup E_i(1)$ where each of $E_i(0)$ and $E_i(1)$ is a disk, $k_1\circ u_1\circ v_1\circ w_1(E_i(0))\subset\text{Int } R(V_r'(k))$ and $k_1\circ u_1\circ v_1\circ w_1(E_i(1))\subset\text{Int } R(V_r''(k))$, and

(24) if $I(S_i)=(r-1,k)$ then

$$k_1\circ u_1\circ v_1\circ w_1(D_i(1)\cup\cdots\cup D_i(q_i))\subset \text{Int } R(V_{r-1}'(k)),$$

$$k_1\circ u_1\circ v_1\circ w_1(B_i(1)\cup\cdots\cup B_i(q_i-1))\subset \text{Int } R(V_{r-1}''(k)),\quad\text{and}$$

$$k_1\circ u_1\circ v_1\circ w_1(P_i)\subset \text{Int } [R(V_{r-1}''(k))\cup R(V_r'(k))].$$

Now if $I(S_i)=(r,k)$, the position of $u_1\circ v_1\circ w_1(S_i)$ is described by (21). Thus we may choose arcs $B_i(1),\ldots,$ and $B_i(q_i-1)$ in P_i such that if $1\leqq l<q_i$ then Int $B_i(l)\subset$ Int P_i and $B_i(l)$ has one endpoint on Bd $D_i(l)$ and the other on Bd $D_i(l+1)$. Using the techniques of Step 2 again, we construct a push k_t on $S^2\times D^2$ which satisfies (22) and (24) and such that

(25) for all t, $k_t(R(V_r'(k)))\subset R(V_r'(k))$, $k_t(R(V_r''(k)))\subset R(V_r''(k))$, and

$$k_t(R(V_{r-1}''(k)))\subset R(V_{r-1}''(k))\cup R(V_r'(k)),$$

and

(26) if $I(S_i)=(r,k)$ and $1\leqq l<q_i$ then

$$k_1\circ u_1\circ v_1\circ w_1(B_i(l))\subset \text{Int } [R(V_r'(k))\cap R(V_r''(k))].$$

Now if $I(S_i)=(r, k)$ then

$$D_i(1) \cup \cdots \cup D_i(q_i) \cup B_i(1) \cup \cdots \cup B_i(q_i-1)$$

is a connected and simply connected subset of S_i and

$$k_1 \circ u_1 \circ v_1 \circ w_1(D_i(1) \cup \cdots \cup D_i(q_i) \cup B_i(1) \cup \cdots \cup B_i(q_i-1))$$

is contained in Int $R(V_r'(k))$. Hence there exists a disk $E_i(1)$ such that

$$D_i(1) \cup \cdots \cup D_i(q_i) \cup B_i(1) \cup \cdots \cup B_i(q_i-1) \subset E_i(1) \subset S_i$$

and $k_1 \circ u_1 \circ v_1 \circ w_1(E_i(1)) \subset \text{Int } R(V_r'(k))$. Then Step 4 is completed by letting $E_i(0)=S_i-\text{Int } E_i(1)$.

Now we describe what has been accomplished by w_t, v_t, u_t, and k_t:

(27) if $I(S_i)=(j, k)$, $1 \leqq j \leqq r-1$, then $S_i=P_i \cup D_i(1) \cup \cdots \cup D_i(q_i-1)$ where P_i is a disk-with-holes, $D_i(1), \ldots,$ and $D_i(q_i)$ are the components of $S_i-\text{Int } P_i$, $k_1 \circ u_1 \circ v_1 \circ w_1(P_i) \subset \text{Int } [R(V_j''(k)) \cup R(V_{j+1}'(k))]$ and if $1 \leqq l \leqq q_i$,

$$k_1 \circ u_1 \circ v_1 \circ w_1(D_i(l)) \subset \text{Int } R(V_j'(k)).$$

Also there are arcs $B_i(1), \ldots,$ and $B_i(q_i-1)$ such that if $1 \leqq l \leqq q_i$ then Int $B_i(l) \subset \text{Int } P_i$, $B_i(l)$ has one endpoint on Bd $D_i(l)$ and the other on Bd $D_i(l+1)$, and $k_1 \circ u_1 \circ v_1 \circ w_1(B_i(l)) \subset \text{Int } R(V_j''(k))$. Also

(28) if $I(S_i)=(r, k)$ then $S_i=E_i(0) \cup E_i(1)$ where $E_i(0)$ and $E_i(1)$ are disks, $k_1 \circ u_1 \circ v_1 \circ w_1(E_i(0)) \subset \text{Int } R(V_r'(k))$ and $k_1 \circ u_1 \circ v_1 \circ w_1(E_i(1)) \subset \text{Int } R(V_r''(k))$.

Step 5. *A second adjustment in* $R(V_1)$, $R(V_2), \ldots,$ *and* $R(V_{r-1})$. In this final step we construct a push f_t on $S^2 \times D^2$ such that

(29) for each $i=1, \ldots,$ or m, $S_i=E_i(0) \cup E_i(1)$ where $E_i(0)$ and $E_i(1)$ are disks and

(a) if $I(S_i)=(1, k)$ then

$$f_1 \circ k_1 \circ u_1 \circ v_1 \circ w_1(E_i(0)) \subset \text{Int } R(V_1'(k))$$

and

$$f_1 \circ k_1 \circ u_1 \circ v_1 \circ w_1(E_i(1)) \subset \text{Int } [R(V_1''(k)) \cup R(V_2'(k))],$$

(b) if $I(S_i)=(j, k)$, $1<j<r$, then $f_1 \circ k_1 \circ u_1 \circ v_1 \circ w_1(E_i(0))$ is contained in Int $[R(V_{j-1}''(k)) \cup R(V_j'(k))]$ and $f_1 \circ k_1 \circ u_1 \circ v_1 \circ w_1(E_i(1))$ is contained in

$$\text{Int } [R(V_j''(k)) \cup R(V_{j+1}'(k))],$$

and

(c) if $I(S_i)=(r, k)$ then $f_1 \circ k_1 \circ u_1 \circ v_1 \circ w_1(E_i(0))$ is contained in

$$\text{Int } [R(V_{r-1}''(k)) \cup R(V_r'(k))]$$

and $f_1 \circ k_1 \circ u_1 \circ v_1 \circ w_1(E_i(1)) \subset \text{Int } R(V_r''(k))$.

First we construct f_t on V_{r-1}. Now if $I(S_i)=(r-1, k)$ then

$$k_1 \circ u_1 \circ v_1 \circ w_1(D_i(1) \cup \cdots \cup D_i(q_i)) \subset \text{Int } R(V_{r-1}'(k))$$

and

$$k_1 \circ u_1 \circ v_1 \circ w_1(B_i(1) \cup \cdots \cup B_i(q_i-1)) \subset \text{Int } R(V_{r-1}''(k)).$$

Hence, using the techniques of Step 2, we may construct a push f_t^{r-1} on $S^2 \times D^2$ such that

(30) $f_t^{r-1} = \text{id}$ outside $R(V''_{r-1}(k))$,

(31) for all t and $k=1$ or 2, $f_t^{r-1}(R(V'_{r-1}(k))) \subset R(V'_{r-1}(k))$,

$$f_t^{r-1}(R(V''_{r-1}(k))) \subset R(V''_{r-1}(k)),$$

and

$$f_t^{r-1}(R(V'_r(k))) \subset R(V''_{r-1}(k)) \cup R(V'_r(k)),$$

and

(32) if $I(S_i) = (r-1, k)$ and $1 \leqq l < q_i$ then

$$f_1^{r-1} \circ k_1 \circ u_1 \circ v_1 \circ w_1(B_i(l)) \subset \text{Int}\,[R(V'_{r-1}(k)) \cap R(V''_{r-1}(k))].$$

Then for $I(S_i) = (r-1, k)$,

$$f_1^{r-1} \circ k_1 \circ u_1 \circ v_1 \circ w_1(D_i(1) \cup \cdots \cup D_i(q_i) \cup B_i(1) \cup \cdots \cup B_i(q_i - 1))$$

is contained in Int $R(V'_{r-1}(k))$. Hence there is a disk $E_i(0)$ such that

$$D_i(1) \cup \cdots \cup D_i(q_i) \cup B_i(1) \cup \cdots \cup B_i(q_i - 1) \subset E_i(0) \subset S_i$$

and

(33) $f_1^{r-1} \circ k_1 \circ u_1 \circ v_1 \circ w_1(E_i(0)) \subset \text{Int}\, R(V'_r(k))$.

Let $E_i(1) = S_i - \text{Int}\, E_i(0)$. Then $E_i(1) \subset P_i$ so

(34) $f_1^{r-1} \circ k_1 \circ u_1 \circ v_1 \circ w_1(E_i(1)) \subset \text{Int}\,[R(V''_{r-1}(k)) \cup R(V'_r(k))]$.

Also, (31) and (28) imply (29(c)) is satisfied if f_1 is replaced by f_1^{r-1}. Similarly there is a push f_t^{r-2} on $S^2 \times D^2$ such that

(35) $f_t^{r-2} = \text{id}$ outside $R(V''_{r-2}(k))$,

(36) for all t and $k=1$ or 2,

$$f_t^{r-2}(R(V'_{r-2}(k))) \subset R(V'_{r-2}(k)), \quad f_t^{r-2}(R(V''_{r-2}(k))) \subset R(V''_{r-2}(k)),$$

and $f_t^{r-2}(R(V'_{r-1}(k))) \subset R(V''_{r-2}(k)) \cup R(V'_{r-1}(k))$, and

(37) if $I(S_i) = (r-2, k)$ and $1 \leqq l < q_i$ then

$$f_1^{r-2} \circ k_1 \circ u_1 \circ v_1 \circ w_1(B_i(l)) \subset \text{Int}\,[R(V'_{r-2}(k)) \cap R(V''_{r-1}(k))].$$

Then if $I(S_i) = (r-2, k)$,

$$f_1^{r-2} \circ k_1 \circ u_1 \circ v_1 \circ w_1(D_i(1) \cup \cdots \cup D_i(q_i) \cup B_i(1) \cup \cdots \cup B_i(q_i - 1))$$

is contained in Int $R(V'_{r-2}(k))$. Hence there is a disk $E_i(0)$ such that

$$D_i(1) \cup \cdots \cup D_i(q_i) \cup B_i(1) \cup \cdots \cup B_i(q_i - 1) \subset E_i(0) \subset S_i$$

and

(38) $f_1^{r-2} \circ k_1 \circ u_1 \circ v_1 \circ w_1(E_i(0)) \subset \text{Int } R(V'_{r-2}(k))$.

As above, let $E_i(1) = S_i - \text{Int } E_i(0)$. By (27) and (37),

(39) $f_1^{r-2} \circ k_1 \circ u_1 \circ v_1 \circ w_1(E_i(1))$ is contained in Int $[R(V''_{r-2}(k)) \cup R(V'_{r-1}(k))]$.

Also, (36), (33) and (34) imply (29(b)) is satisfied for $j=r-1$, if f_1 is replaced by $f_1^{r-2} \circ f_1^{r-1}$.

We continue in this manner to construct a push f_t^{r-3} on $S^2 \times D^2$ such that $f_t^{r-3} = \text{id}$ outside $R(V'_{r-2})$, so that (29(b)) is satisfied for $j=r-2$ if f_1 is replaced by $f_1^{r-3} \circ f_1^{r-2} \circ f_1^{r-1}$ and so that if $I(S_i) = (r-3, k)$ then S_i is a union of disks $E_i(0)$ and $E_i(1)$ which satisfy properties similar to (38) and (39). Then in succession we construct $f_t^{r-4}, \ldots,$ and f_t^1. We finish Step 5 by letting $f_t = f_t^1 * f_t^2 * \cdots * f_t^{r-1}$. We finish the lemma by letting $h_t = f_t * k_t * u_t * v_t * w_t$.

Proof of Proposition $(r, 1)$. Since we have already proved Proposition (1, 1), we assume $r > 1$ and that Proposition $(r-1, 1)$ is true. Hence we have

$$0 = t_0 < t_1 < \cdots < t_{r+1} = 2 \quad \text{and} \quad 0 = a_0 < a_1 < a_2 = 2\pi.$$

Choose numbers $s_0, \ldots,$ and s_r such that $s_0 = t_0$, $s_r = t_{r+1}$ and $t_i < s_i < t_{i+1}$ if $1 \leq i < r$. Then by Proposition $(r-1, 1)$, there is a push w_t on $S^2 \times D^2$ and an integer n' such that for each stage n' index α, $w_1(F_0(X_\alpha))$ is contained in

$$\text{Int } R_e(W^*(s_{i-1}, s_i, a_{j-1}, a_j))$$

for some $i = 1, \ldots,$ or r and $j = 1$ or 2. Then, by Lemma 4, there exists a push v_t on $S^2 \times D^2$ such that for each stage n' index α, $S_\alpha = E_\alpha(0) \cup E_\alpha(1)$ where $E_\alpha(1)$ and $E_\alpha(2)$ are disks and such that if $l = 0$ or 1 then $v_1 \circ w_1 \circ F_0(E(l) \times \{0\})$ is contained in Int $R_e(W^*(t_{j-1}, t_j, a_{k-1}, a_k))$ for some $j = 1, \ldots,$ or $r+1$ and $k = 1$ or 2. For each stage n' index α choose a disk $\tilde{D}_\alpha$ in Int D_α and neighborhoods $R(E_\alpha(0))$ and $R(E_\alpha(1))$ of $E_\alpha(0)$ and $E_\alpha(1)$ on S_α such that if $l = 0$ or 1 then

$$v_1 \circ w_1 \circ F_0(E_\alpha(l) \times \tilde{D}_\alpha) \subset \text{Int } R_e(W^*(t_{j-1}, t_j, a_{k-1}, a_k))$$

for the appropriate j and k. Then, by Lemma 2, for each stage n' index α there exists a push f_t^α on $v_1 \circ w_1 \circ F_0(S_\alpha \times D_\alpha)$ such that if β is a stage $(n'+1)$ index and $X_\beta \subset X_\alpha$ then there exists $l = 0$ or 1 such that

$$f_1 \circ v_1 \circ w_1 \circ F_0(S_\beta \times \{0\}) \subset v_1 \circ w_1 \circ F_0(E_\alpha(l) \times \tilde{D}_\alpha).$$

Let u_t be the push on $S^2 \times D^2$ such that

$$u_t | v_1 \circ w_1 \circ F_0(S_\alpha \times D_\alpha) = f_t^\alpha \quad \text{and} \quad u_t = \text{id elsewhere.}$$

Then for each stage $(n'+1)$ index β

$$u_1 \circ v_1 \circ w_1 \circ F_0(S_\beta \times \{0\}) \subset \text{Int } R_e(W^*(t_{j-1}, t_j, a_{k-1}, a_k))$$

for some j and k. Then for each stage $(n'+1)$ index β there is a push g_t^β on $u_1 \circ v_1 \circ w_1 \circ F_0(S_\beta \times D_\beta)$ such that

$$g_1^\beta(u_1 \circ v_1 \circ w_1 \circ F_0(X_\beta \cap M_{n'+2})) \subset \text{Int } R_e(W^*(t_{j-1}, t_j, a_{k-1}, a_k)).$$

Let k_t be the push on $S^2 \times D^2$ such that

$$k_t|u_1 \circ v_1 \circ w_1 \circ F_0(X_\beta) = g_t^\beta \quad \text{and} \quad k_t = \text{id elsewhere.}$$

Finally let $h_t = k_t * u_t * v_t * w_t$ and $n = n' + 2$.

6. **Proposition** (r, s).

LEMMA 5. *Suppose* $S^2 = Z_0 \cup Z_1 \cup \cdots \cup Z_q$ *where* Z_0 *is a disk-with-holes and* $Z_1, \ldots,$ *and* Z_q *are the components of* $S^2 - \text{Int } Z_0$, $\tilde{D}^2 \subset \text{Int } D^2$ *and* $\varepsilon > 0$. *Then there is a push* h_t *on* $S^2 \times D^2$ *and an integer n such that for each stage n index* α *there exists* $j = 0, \ldots,$ *or q such that* $h_1(F(X_\alpha)) \subset \text{Int } (N_\varepsilon(Z_j) \times \tilde{D}^2)$.

Proof. It is easy to show there exist numbers $t_0, t_1, \ldots,$ and t_{r+1} and a push g_t on S^2 such that $0 = t_0 < t_1 < \cdots < t_r < t_{r+1} = 2$ and

$$\text{Bd } Z_0 \subset g_1[\bigcup \{C(t_i) : i = 0, \ldots, \text{or } r+1\} \cup M(0) \cup M(\pi)].$$

Then we can choose $e > 0$ such that if $1 \leqq j \leqq r+1$ and $k = 1$ or 2 then $g_1(R_e(W(t_{j-1}, t_j, (k-1)\pi, k\pi))) \subset N_\varepsilon(Z_i)$ for some $i = 0, \ldots,$ or q. Hence Lemma 5 follows from Proposition $(r, 1)$.

LEMMA 6. *Suppose* $0 = t_0 < t_1 < \cdots < t_{r+1} = 2$ *and* $0 = b_0 < b_1 < \cdots < b_s = 2\pi$. *Let* $a_k = b_k$ *for* $0 \leqq k \leqq s-2$ *and suppose* $b_{s-2} < a_{s-1} < b_{s-1} < a_s < b_s = a_{s+1} = 2\pi$ *and* $e > 0$. *Suppose also that* $S_1, \ldots,$ *and* S_m *are mutually disjoint polyhedral 2-spheres in* $\text{Int } (S^2 \times D^2)$ *such that if* $1 \leqq i \leqq m$ *then there exists* $j = 1, \ldots,$ *or s and* $k = 1, \ldots,$ *or s such that* $S_i \subset \text{Int } R_e(W^*(t_{j-1}, t_j, b_{k-1}, b_k))$. *Then there exists a push* h_t *on* $S^2 \times D^2$ *such that* $h_t = \text{id}$ *outside* $R_e(Z(a_{s-1}, a_s))$ *and such that for each i either*

(1) $h_1(S_i) \subset \text{Int } R_e(W^*(t_{j-1}, t_j, a_{k-1}, a_k))$ *for some* $j = 1, \ldots,$ *or* $r+1$ *and some* $k = 1, \ldots,$ *or* $s+1$ *or*

(2) S_i *contains a disk-with-holes* P_i *such that there exists* $j = 1, \ldots,$ *or* $s+1$ *and* $k = 1, \ldots,$ *or* $s+1$ *such that either*

(a) $h_1(P_i) \subset \text{Int } R_e(W^*(t_{j-1}, t_j, a_{k-2}, a_{k-1}))$ *and*

$$h_1(S_i - \text{Int } P_i) \subset \text{Int } R_e(W^*(t_{j-1}, t_j, a_{k-1}, a_k))$$

or

(b) $h_1(P_i) \subset \text{Int } R_e(W^*(t_{j-1}, t_j, a_{k-1}, a_k))$ *and*

$$h_1(S_i - \text{Int } P_i) \subset \text{Int } R_e(W^*(t_{j-1}, t_j, a_{k-2}, a_{k-1})).$$

Proof of Lemma 6. Lemma 6 is proved by the arc-pulling techniques of Step 2 of Lemma 4.

Proof of Proposition (r, s). We suppose $r \geqq 1, s > 1$ and Proposition $(r, s-1)$ is true. We have $0 = t_0 < t_1 < \cdots < t_{r+1} = 2$ and $0 = a_0 < a_1 < \cdots < a_{s+1} = 2\pi$. Let $b_k = a_k$ for $0 \leqq k \leqq s-2$ and choose numbers b_{s-1} and b_s such that $a_{s-1} < b_{s-1} < a_s = b_s = 2\pi$. By Proposition $(r, s-1)$ there is a push w_t on $S^2 \times D^2$ and an integer n' such that, for each stage n' index α, $w_1 \circ F_0(X_\alpha) \subset \text{Int } R_e(W^*(t_{i-1}, t_i, b_{j-1}, b_j))$ for some $i = 1, \ldots,$ or $r+1$ and some $j = 1, \ldots,$ or s. Then, using Lemma 5 and Lemma 6, we may find a push v_t on $S^2 \times D^2$ and an integer $n'' > n'$ such that if β is a stage n'' index then $v_1 \circ w_1(S_\beta \times \{0\}) \subset \text{Int } R_e(W^*(t_{i-1}, t_i, a_{j-1}, a_j))$ for some $i=1, \ldots,$ or $r+1$ and some $j=1, \ldots,$ or $s+1$. Finally there is a push u_t on $S^2 \times D^2$ such that $u_t = \text{id}$ outside $v_1 \circ w_1 \circ F_0(M_{n''})$ and such that if γ is a stage $(n''+1)$ index then

$$u_1 \circ v_1 \circ w_1 \circ F_0(X_\gamma) \subset \text{Int } R_e(W^*(t_{i-1}, t_i, a_{j-1}, a_j))$$

for the appropriate i and j. Then we let $h_t = u_t * v_t * w_t$ and $n = n''+1$.

References

1. S. Armentrout and R. H. Bing, *A toroidal decomposition of* E^3, Fund. Math. **60** (1967), 81–87. MR **34** #6741.

2. R. H. Bing, *Upper semicontinuous decompositions of* E^3, Ann. of Math. (2) **65** (1957), 363–374. MR **19**, 1187.

3. ———, *Point-like decompositions of* E^3, Fund. Math. **50** (1961/62), 431–453. MR 25 #560.

4. J. F. P. Hudson, *Piecewise linear topology*, Benjamin, New York, 1969.

5. L. L. Lininger, *Actions on* S^n (to appear).

6. R. B. Sher, *Toroidal decompositions of* E^3, Fund. Math. **61** (1967/68), 225–241. MR **37** #905.

Kent State University,
Kent, Ohio 44240